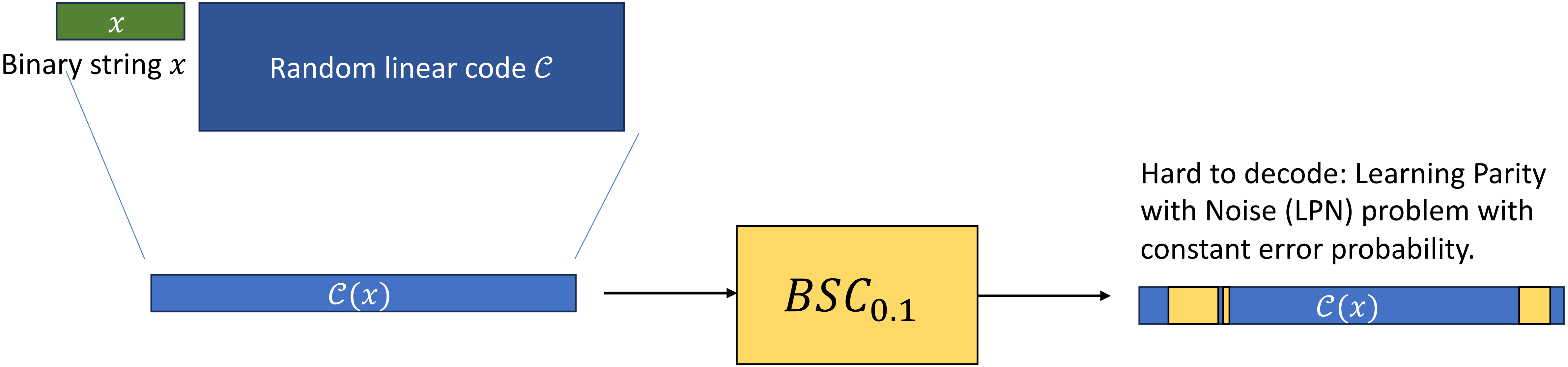
Computational Wiretap Coding from Indistinguishability Obfuscation

Yuval Ishai (Technion), Aayush Jain (CMU), Paul Lou (UCLA),
Amit Sahai (UCLA), Mark Zhandry (NTT Research)

Teaser: Interesting special case of the general wiretap problem

Teaser: Curious Coding Theory Question



Hard to decode: Learning Parity with Noise (LPN) problem with constant error probability.

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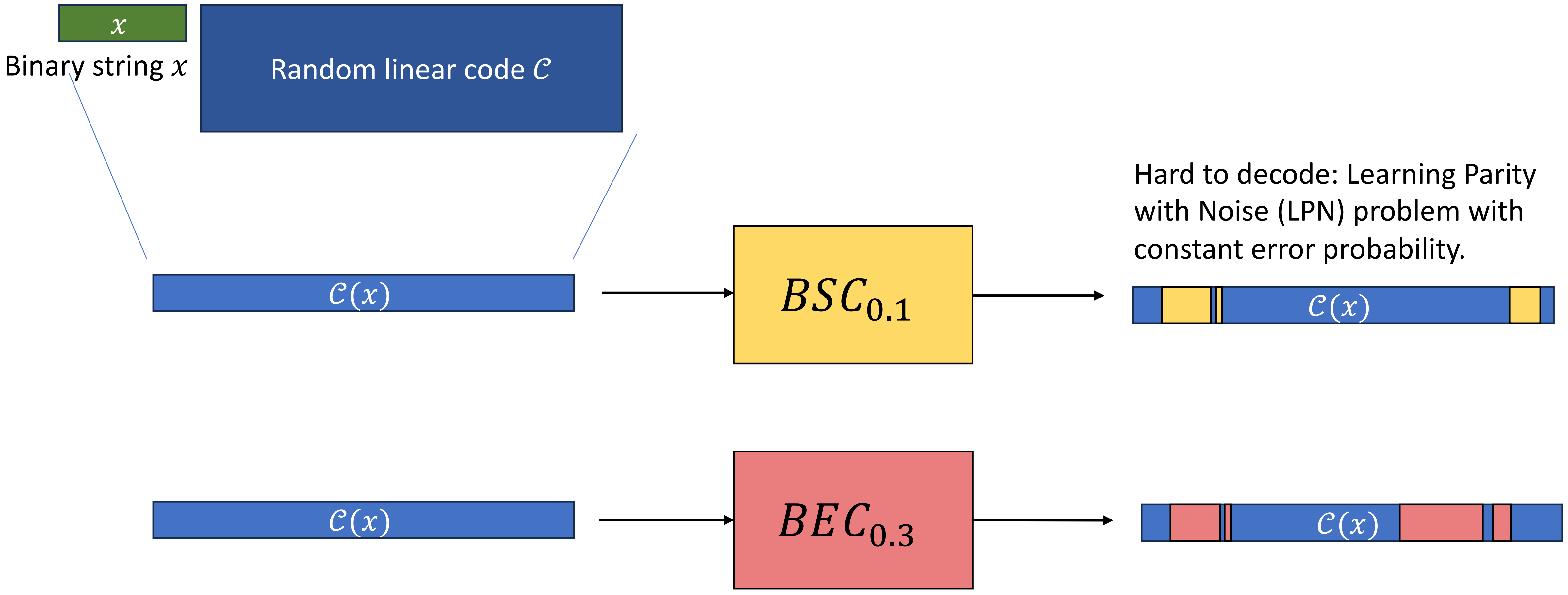

For the right choice of parameters, Gaussian elimination recovers x.

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Do there exist error-correcting codes that satisfy the following?

1. **Easy to decode from 0.1 bitflip error rate. [LDPC, BCH, etc.]**
2. **Computationally hard to decode from 0.3 erasure rate. [Linear codes fail]**

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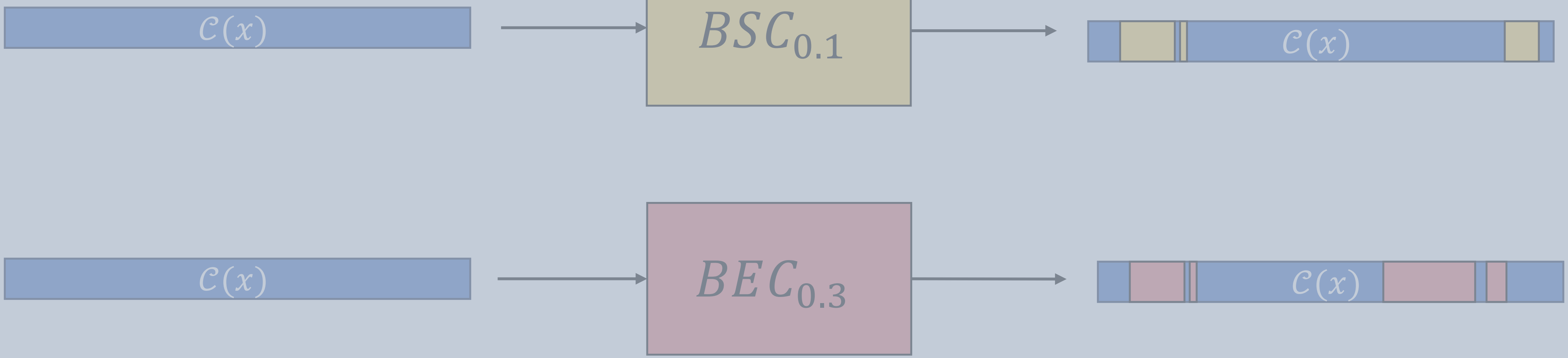

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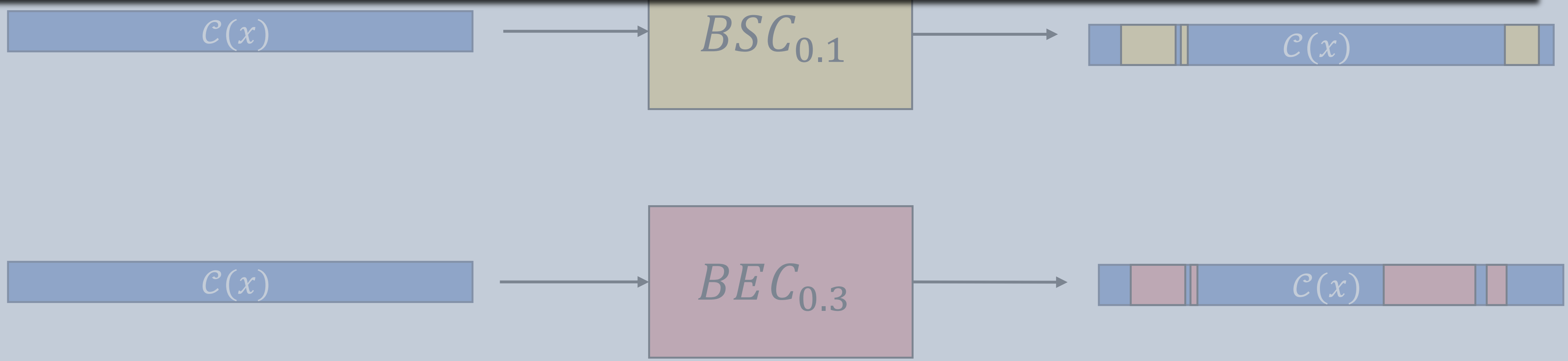
Hard to decode: Learning Parity

Until last year, no such codes known to satisfy both.



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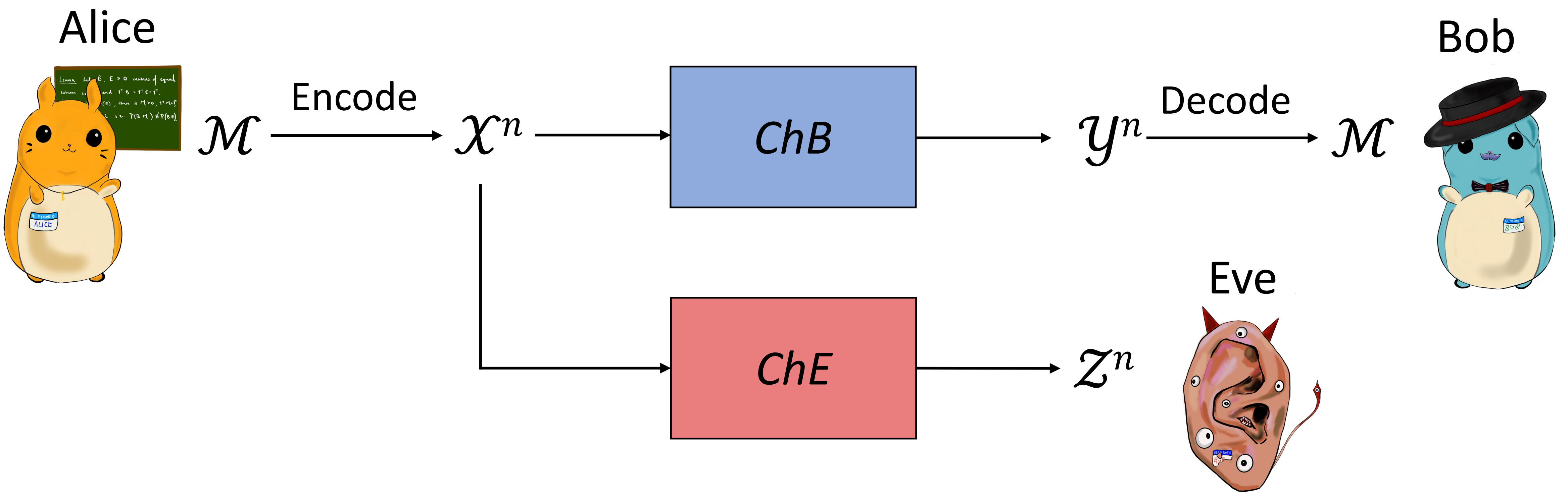
Ishai, Korb, Lou, Sahai '22: Yes*, in the ideal obfuscation model (or non-standard VBB obfuscation assumptions)!

This Work: Yes*, assuming standard hardness assumptions!

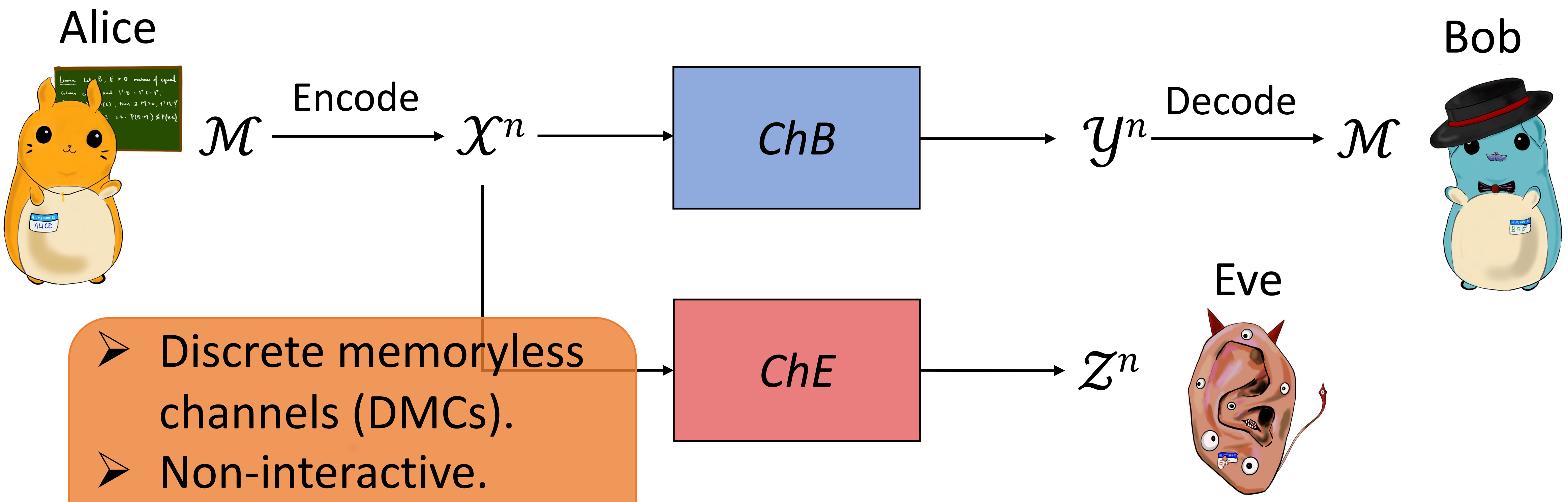
For the right choice of parameters, Gaussian elimination recovers x.

General Setting: Wiretap Channel [Wyn75]



Goal: Alice wants to send a message to Bob without Eve learning it.

More General Setting:
Wiretap Channel [Wyn75]



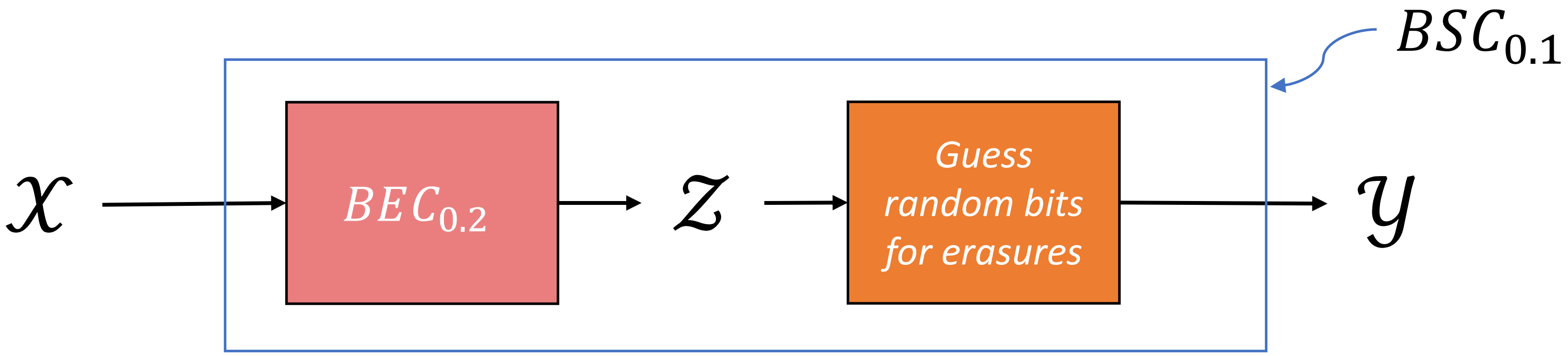
- Discrete memoryless channels (DMCs).
- Non-interactive.
- No shared secrets.

Goal: Alice wants to send a message to Bob without Eve learning it.

For what pairs of channels do wiretap coding schemes exist?

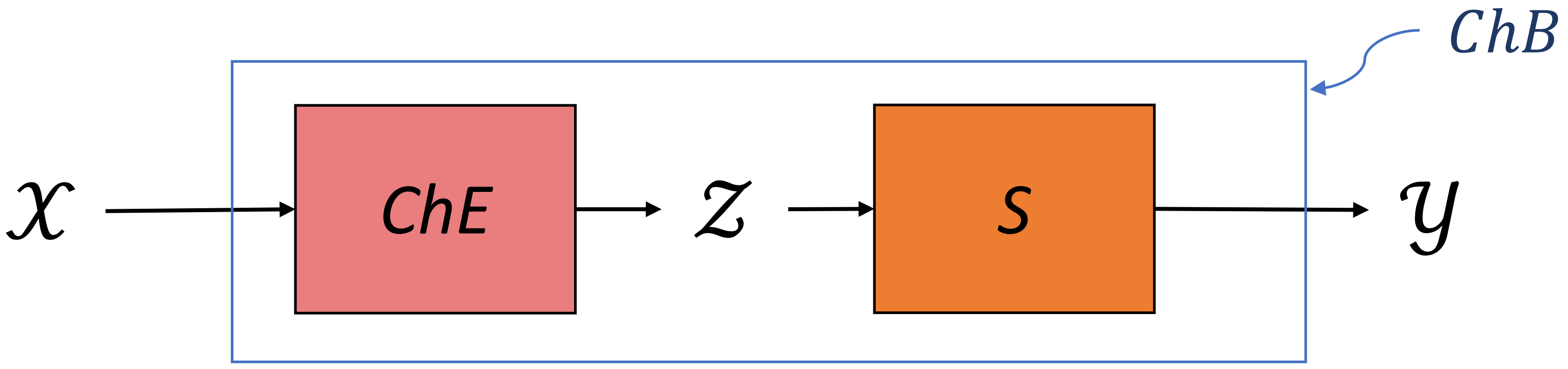
Intuitive Impossibility for Degraded Pairs

Impossible for channel pair $(BSC_{0.1}, BEC_{0.2})$. Eve can perfectly simulate $BSC_{0.1}$'s output distribution using an output of $BEC_{0.2}$.



Intuitive Impossibility for Degraded Pairs

Impossible for any channel pair (ChB, ChE) where Eve can perfectly simulate ChB's output distribution using an output of ChE.



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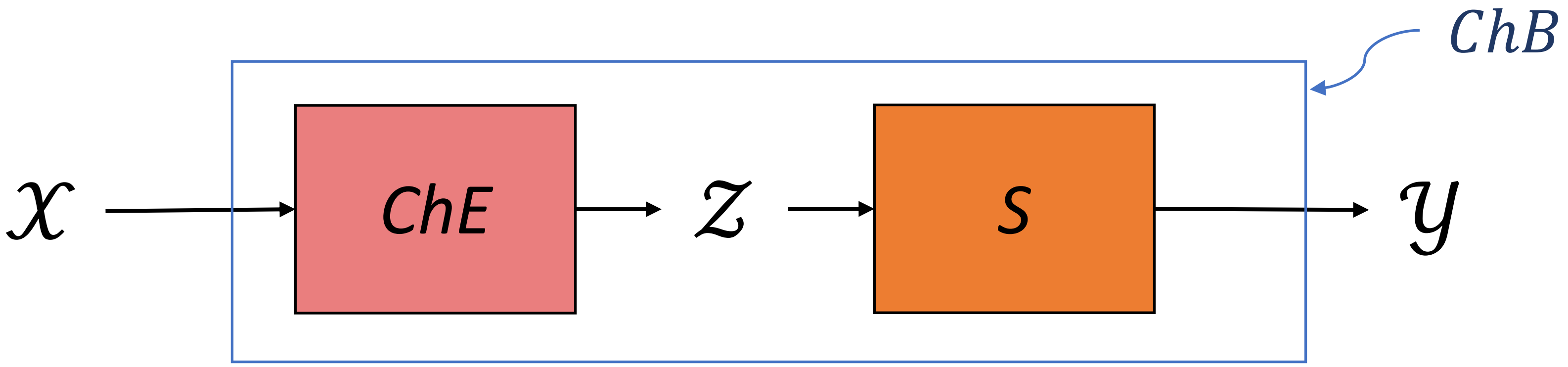
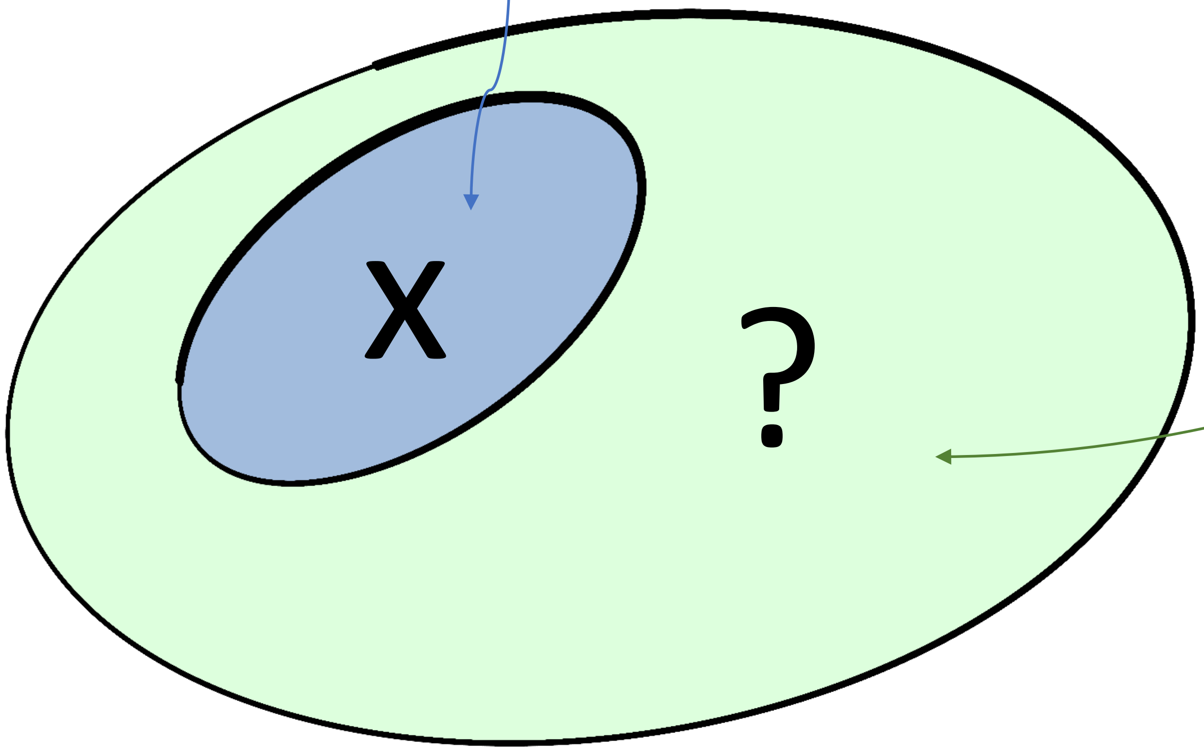


Degradation: ChB is a degradation of ChE if and only if Eve can perfectly simulate ChB using ChE.

Existence of Wiretap Coding Schemes

None for (ChB, ChE) where ChB is a degradation of ChE.

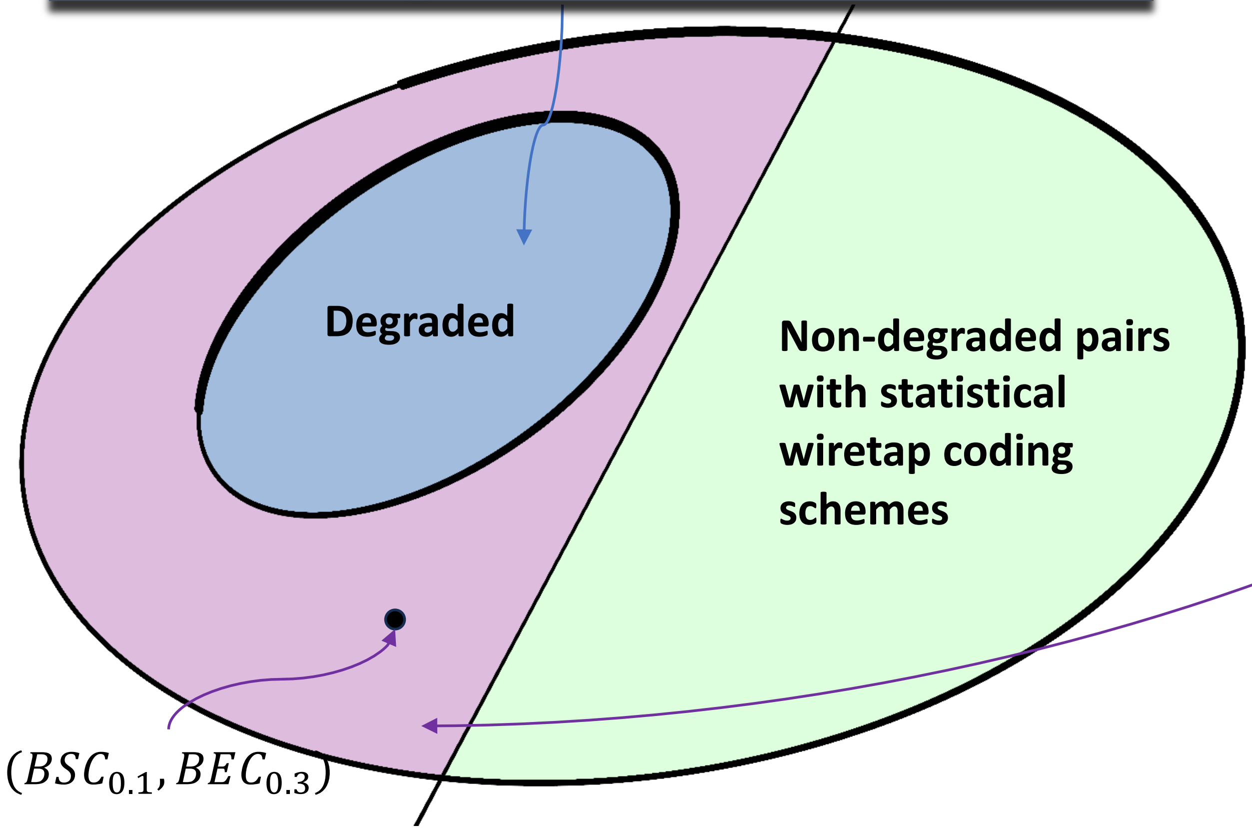
Do there exist wiretap coding schemes for non-degraded channel pairs (ChB, ChE)?



Existence of Wiretap Coding Schemes

None for (ChB, ChE) where ChB is a degradation of ChE.

Csiszár, Korner '78: There are non-degraded channel pairs that do not have statistical wiretap coding schemes.



Existence of Wiretap Coding Schemes

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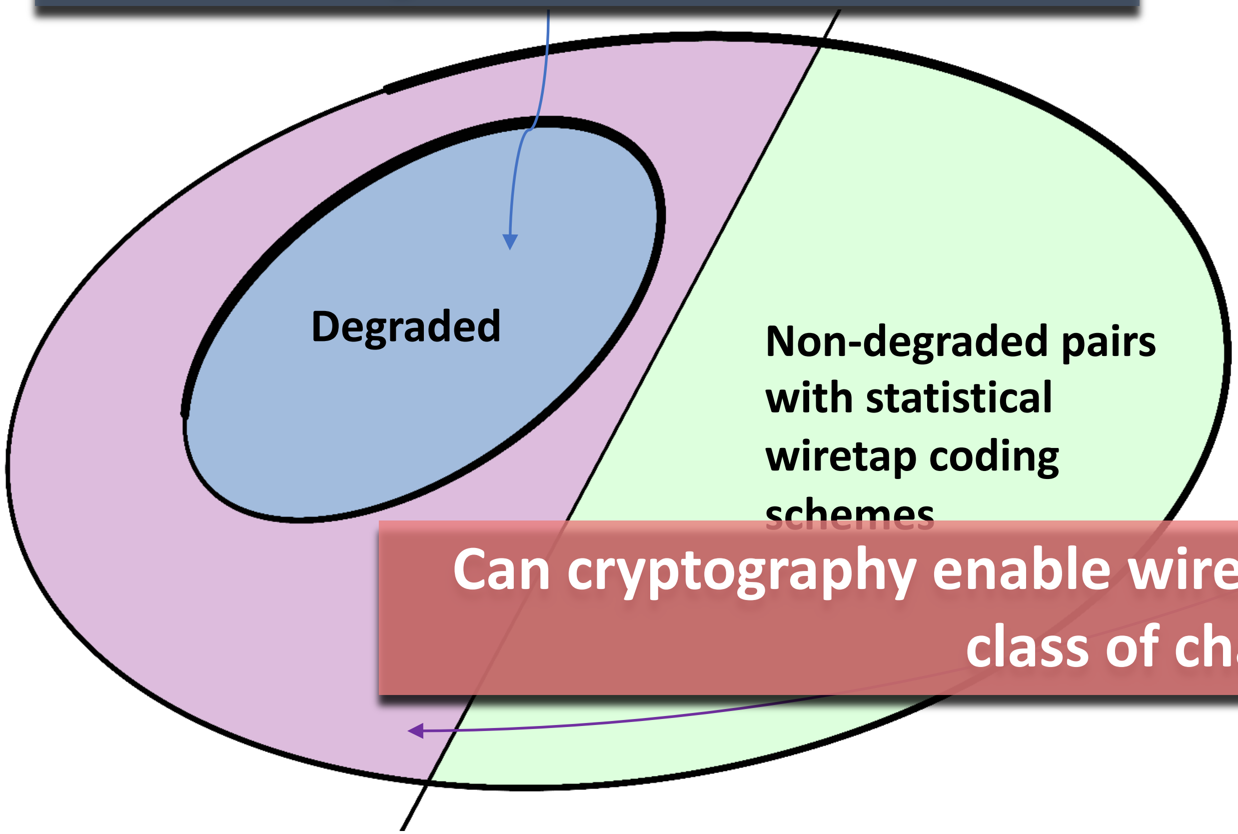
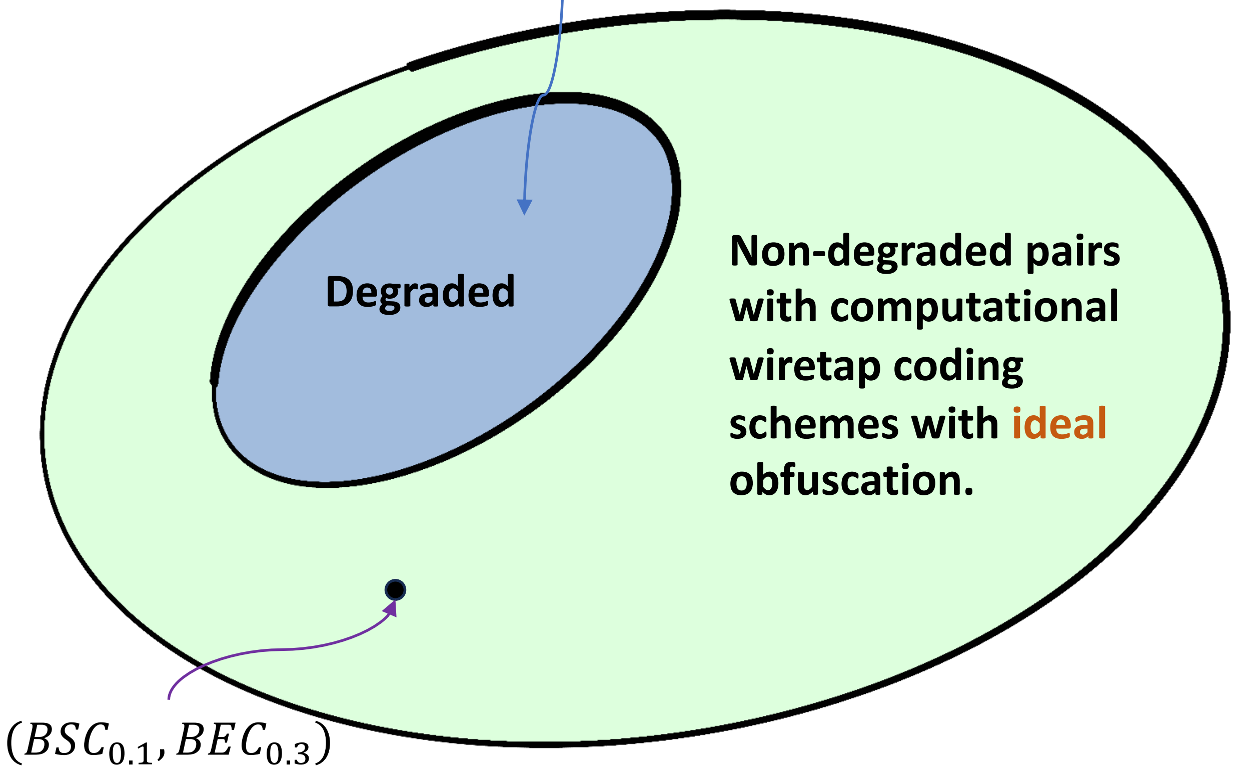


Can cryptography enable wiretap coding schemes for a larger class of channel pairs?

Existence of Wiretap Coding Schemes

None for (ChB, ChE) where ChB is a degradation of ChE.



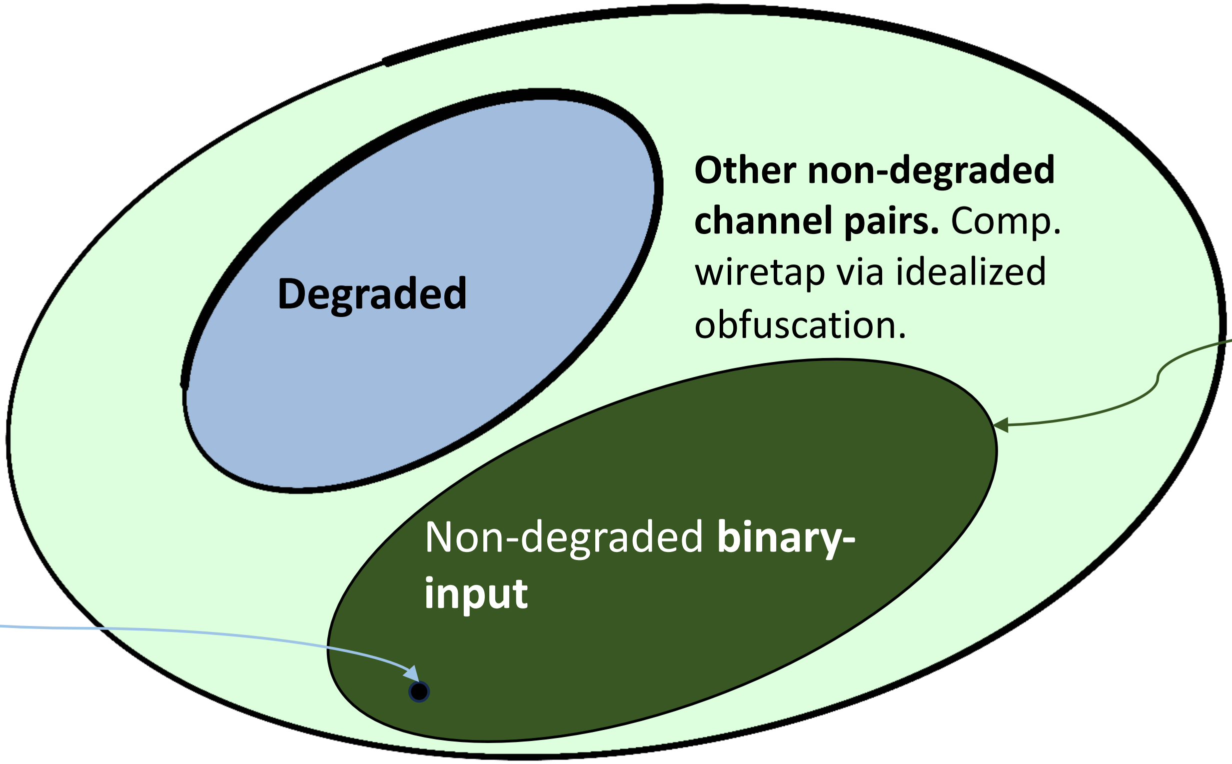
Csiszár, Korner '78: There are non-degraded channel pairs that do not have statistical wiretap coding schemes.

Ishai, Korb, Lou, Sahai '22: There exists a computational wiretap coding scheme for all non-degraded channel pairs in **the Ideal Obfuscation Model (or non-std. VBB obfuscation)**.

Can we obtain computational wiretap coding schemes from standard assumptions?

Our Main Result: YES

Theorem: Assuming the existence of indistinguishability obfuscation (iO) and injective PRGs, there exists a computational wiretap coding scheme for any pair of non-degraded **binary-input** channels (ChB, ChE).



Non-degraded binary-input channel pairs with computational wiretap coding schemes from **standard assumptions**

Solves* the teaser:

$(BSC_{0.1}, BEC_{0.3})$

Our Techniques

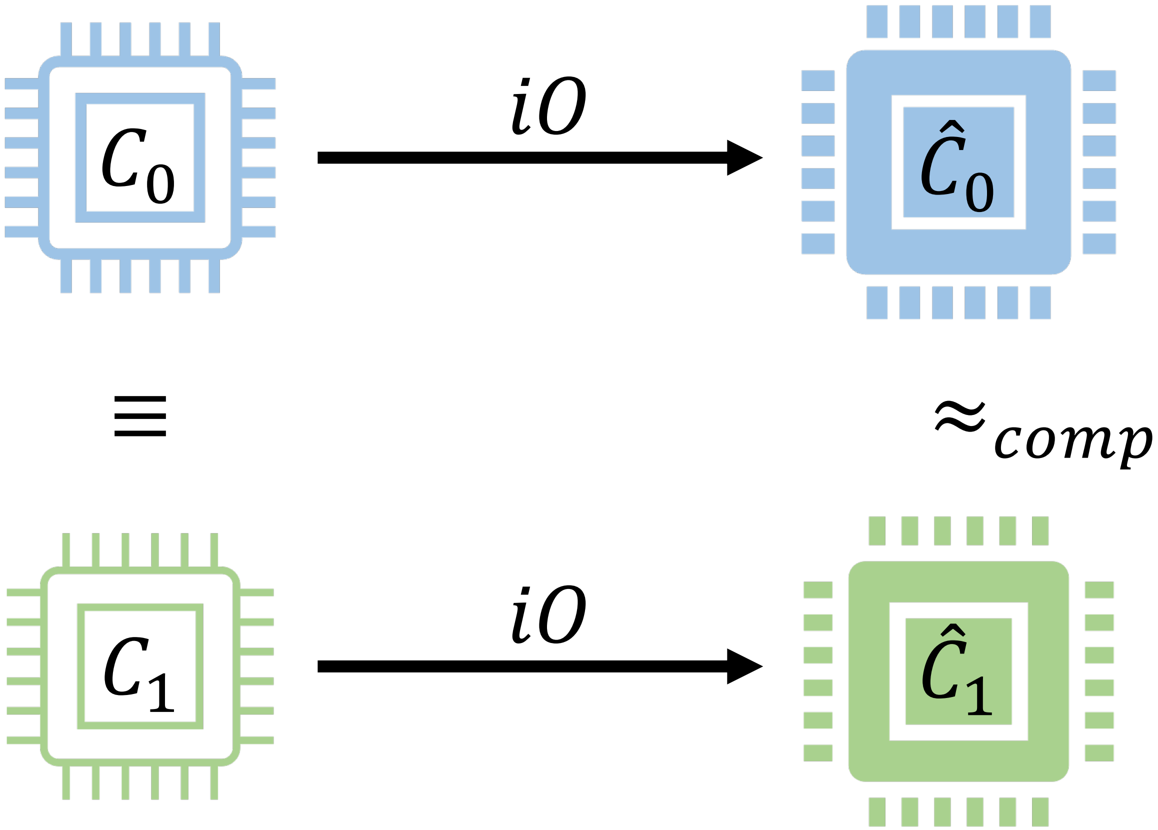
1. Using iO and injective PRGs, we construct a Hamming ball obfuscator.
   - Construction uses a new gadget: PRG with Self-Correction.
   - Using this, we build computational wiretap coding schemes for binary asymmetric channels (BAC) and binary asymmetric erasure channels (BAEC).
2. We introduce a polytope characterization of degradation.
   - Using this polytope characterization, we reduce the problem of constructing a computational wiretap coding scheme for any non-degraded binary-input channel pair to constructing one for (BAC, BAEC).

Focus of this talk:
A computational wiretap coding scheme from $i\mathcal{O}$ for ($ChB = BSC_{0.1}$, $ChE = BEC_{0.3}$)

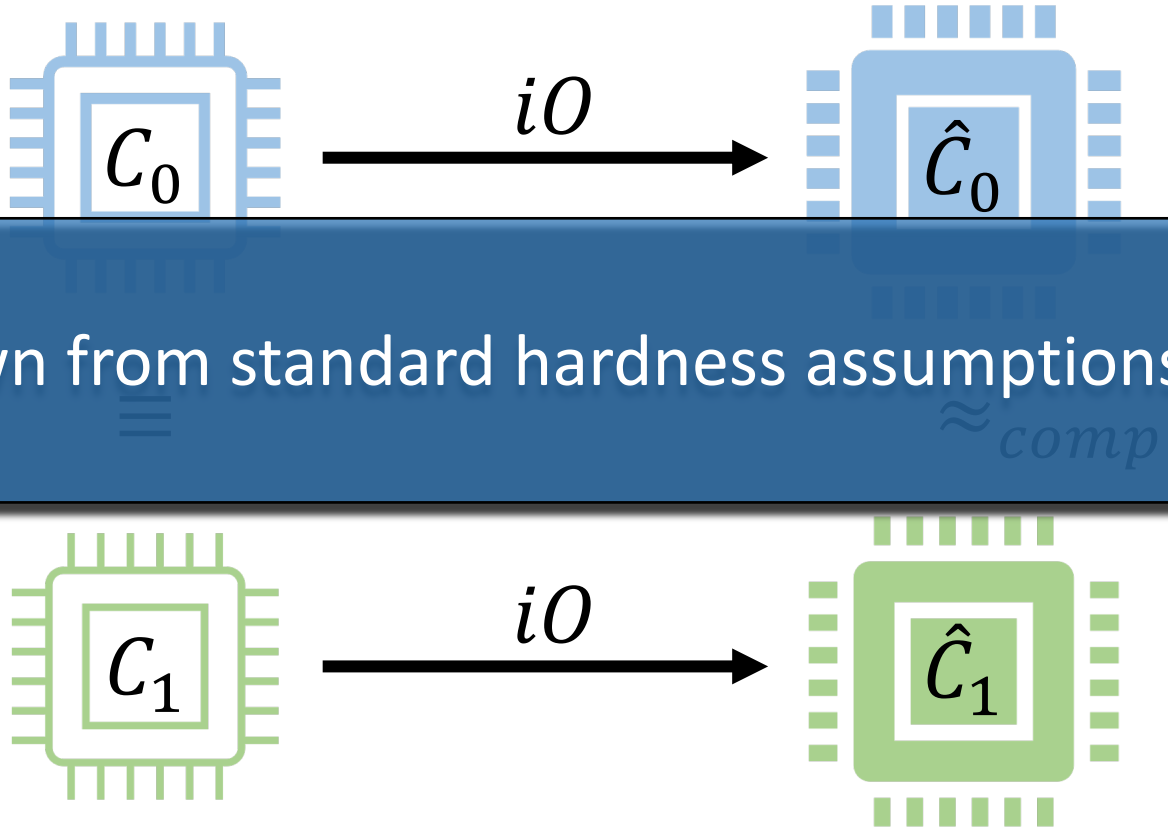
*Construction idea easily extends to the non-degraded (BAC, BAEC) setting.

**See paper or slide appendix for extension to all non-degraded binary-input.

Indistinguishability Obfuscation ($i\mathcal{O}$) [BGIRSVY01]



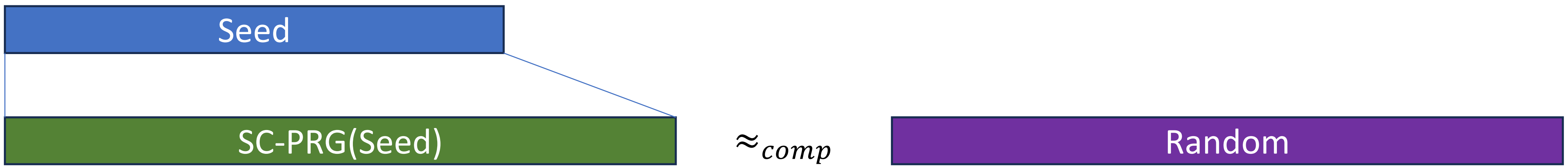
Indistinguishability Obfuscation ($i\mathcal{O}$) [BGIRSVY01]



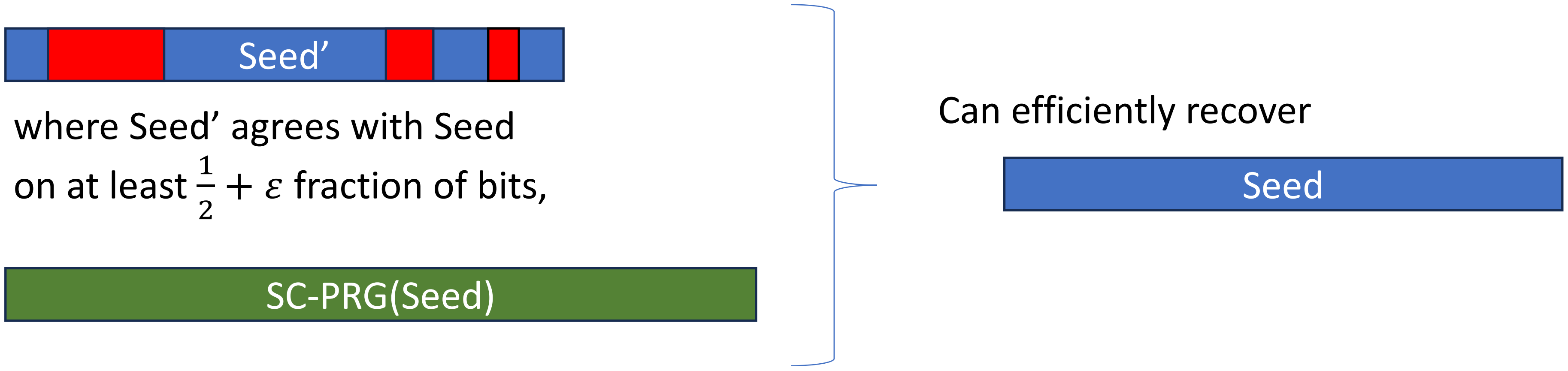
Now known from standard hardness assumptions !! [JLS21]

New Gadget:
PRG with Self-Correction (SCPRG)

1. Polynomial Stretch & Pseudorandomness



2. ε-Self-Correction



New Gadget:
PRG with Self-Correction (SCPRG)

1. Polynomial Stretch & Pseudorandomness

For this talk, $\varepsilon = \frac{1}{12}$. In general, some constant.

Seed

SC-PRG(Seed) $\approx_{comp}$ Random

2. ε-Self-Correction (recovery works w.h.p. over choices of seeds)

Seed'

where Seed' agrees with Seed on at least $\frac{1}{2} + \varepsilon$ fraction of bits,

SC-PRG(Seed)

Can efficiently recover

Seed

New Gadget:
PRG with Self-Correction (SCPRG)

1. Polynomial Stretch & Pseudorandomness

Seed

SC-PRG(Seed) $\approx_{comp}$ Random

Bogdanov, Qiao ‘12: Goldreich PRG, even with linear stretch, is a SCPRG.

2. ε-Self-Correction

Seed’

where Seed’ agrees with Seed on at least $\frac{1}{2} + \varepsilon$ fraction of bits,

SC-PRG(Seed)

Can efficiently recover

Seed

New Gadget:
PRG with Self-Correction (SCPRG)

1. Polynomial Stretch & Pseudorandomness

Seed

SC-PRG(Seed) $\approx_{comp}$ Random

Our Work: Injective SC-PRG from any injective PRG.

2. ε-Self-Correction

Seed'

where Seed' agrees with Seed on at least $\frac{1}{2}+\varepsilon$ fraction of bits,

SC-PRG(Seed)

Can efficiently recover

Seed

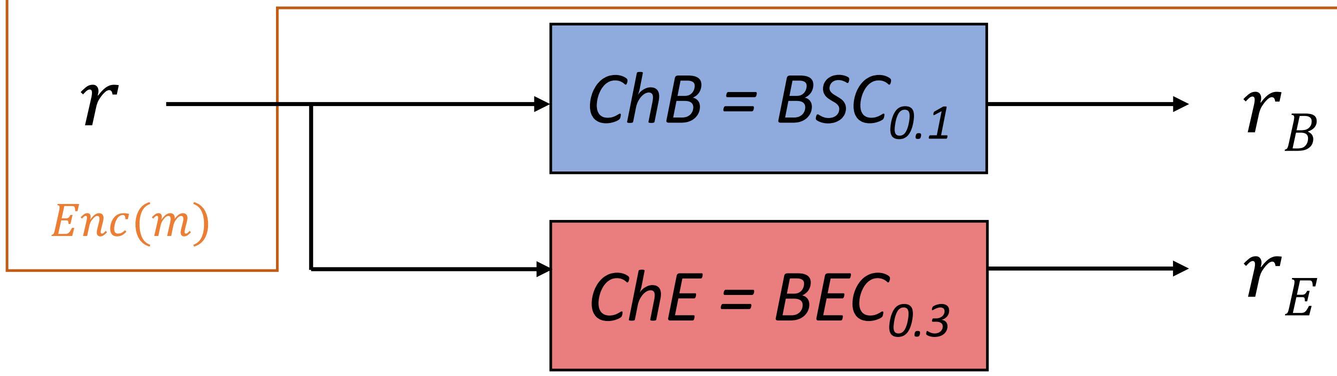
$ChB = BSC_{0.1}$, $ChE = BEC_{0.3}$

Using ideal obfuscation [IKLS22]: Send a uniform random $r \in \{0,1\}^n$ across the wiretap channel. Then, send an obfuscation of f_r, encoded to Bob's channel.

$f_r(r')$:

- If $\Delta(r', r) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

Correctness:
$f_r(r_B) = m$ with high probability



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- Output $\perp$ otherwise.

iO

Correctness:
$f_r(r_B) = m$ with high probability

r

$Enc(m)$

Eve's best guess for r' has ≈ 0.15 error rate.

If we were using an ideal obfuscation, then r and m are hidden.

$ChB = BSC_{0.1}$, $ChE = BEC_{0.3}$

Construction: Send a uniform random $r \in \{0,1\}^n$ across the wiretap channel. Then, send an iO of f_r, encoded to Bob's channel.

$f_r(r')$:

- If $\Delta(r', r) < 0.1n + n^{0.9}$ output m
- Output $\bot$ otherwise.

Correctness:
$f_r(r_B) = m$ with high probability

r

$Enc(m)$

Security: Why does $iO(f_r)$ hide m or r?

Security: What Does Eve See?

Eve sees:

r_E = ⊥010⊥1011⊥

Eve does not know:

r = 1010010110

$f_r(r')$:

- If $\Delta(r', r) < 0.1n + n^{0.9}$ output m
- Output ⊥ otherwise.

Security: What Does Eve See?

Eve sees:

$r_E = \perp 010 \perp 1011 \perp$

Eve does not know:

$r = 1010010110$

Goal: Use a hybrid argument to show that this circuit is indistinguishable from the null circuit.

Problem: There are **exponentially** many points in the Hamming ball!

$f_r(r')$:

- If $\Delta(r', r) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

Security: What Does Eve See?

Eve sees:

$r_E = \bot 010 \bot 1011 \bot$

Eve does not know:

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$f_r(r')$:

- If $\Delta(r', r) < 0.1n + n^{0.9}$ output m
- Output $\bot$ otherwise.

Critical observation: In intermediate hybrids, this circuit can depend on the actual received string r_E.

Security: What Does Eve See?

Eve sees:

$\boxed{r_E = \perp 010 \perp 1011 \perp}$

$S_\perp = \{1, 5, 10\}$ $S_{0,1} = [10] \setminus S_\perp$

$f_r(r')$:

- If $\Delta(r', r) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

Eve does not know:

$r = 1010010110$

Critical observation: In intermediate hybrids, this circuit can depend on the actual received string r_E.

Security: An Indistinguishable Viewpoint

Eve sees:

$r_E = \perp 010 \perp 1011 \perp$

$S_\perp = \{1, 5, 10\}$ $\quad$ $S_{0,1} = [10] \setminus S_\perp$

Eve does not know:

$r = 1010010110$

Split the hardcoded r into two substrings depending on $S_\perp$

$f^{(1)}(r')$:

Constants: $r_{S_{0,1}}, r_{S_\perp}, S_\perp$.

- If $\Delta(r', r) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

Security: An Indistinguishable Viewpoint

Eve sees:

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$S_\perp = \{1, 5, 10\}$ $\quad S_{0,1} = [10] \setminus S_\perp$

Eve does not know:

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r' is Eve's guess.

$f^{(1)}(r')$:

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Eve sees:

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$S_\perp = \{1, 5, 10\}$ $\quad S_{0,1} = [10] \setminus S_\perp$

Eve does not know:

$r = 1010010110$

Rewrite the Hamming distance condition

$f^{(1)}(r')$:

Constants: $r_{S_{0,1}}, r_{S_\perp}, S_\perp$.

- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta(r'_{S_{0,1}}, r_{S_{0,1}}) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

Security: An Indistinguishable Viewpoint

Eve sees:

r_E = $\perp 010 \perp 1011 \perp$

$S_\perp$ = $\{1, 5, 10\}$ $\quad S_{0,1}$ = $[10] \setminus S_\perp$

Eve does not know:

r = 1010010110

$r'_{S_\perp}, r'_{S_{0,1}}$ are substrings of Eve's guess.

$f^{(1)}(r')$:

Constants: $r_{S_{0,1}}, r_{S_\perp}, S_\perp$

- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta(r'_{S_{0,1}}, r_{S_{0,1}}) < 0.1n + n^{0.9}$ output m
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Security: An Indistinguishable Viewpoint

Eve sees:

$r_E = \perp 010 \perp 1011 \perp$

$S_\perp = \{1, 5, 10\}$ $\quad S_{0,1} = [10] \setminus S_\perp$

Eve does not know:

$r = 1010010110$

$r_{S_\perp}, r_{S_{0,1}}$ are substrings of the sent random string.

$f^{(1)}(r')$:

Constants: $r_{S_{0,1}}, r_{S_\perp}, S_\perp$.

- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta(r'_{S_{0,1}}, r_{S_{0,1}}) < 0.1n + n^{0.9}$ output m
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Eve does not know:

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Functionally Equivalent to $f_r(\cdot)$!!

$f^{(1)}(r')$:

Constants: $r_{S_{0,1}}, r_{S_\perp}, S_\perp$.

- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta(r'_{S_{0,1}}, r_{S_{0,1}}) < 0.1n + n^{0.9}$ output m
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Eve sees:

$r_E = \perp 010 \perp 1011 \perp$

$S_\perp = \{1, 5, 10\}$ $S_{0,1} = [10] \setminus S_\perp$

Eve does not know:

$r = 1010010110$

Eve knows the non-erased coordinates.

$f^{(1)}(r')$:

Constants: $r_{S_{0,1}}, r_{S_\perp}, S_\perp$.

- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta(r'_{S_{0,1}}, r_{S_{0,1}}) < 0.1n + n^{0.9}$ output m
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Security: An Indistinguishable Viewpoint

Eve sees:

$r_E = \perp 010 \perp 1011 \perp$

$S_\perp = \{1, 5, 10\}$

Eve does not know:

$r = 1010010110$

Eve's best strategy is to uniformly guess for $r'_{S_\perp}$.

There are exponentially many guesses that cause the function to output m.

We will compress them into a single branch that can be removed by a hybrid argument.

$f^{(1)}(r')$:

Constants: $r_{S_{0,1}}, r_{S_\perp}, S_\perp$

- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta(r'_{S_{0,1}}, r_{S_{0,1}}) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

Using injective length-tripling SCPRGs

Eve sees:

r_E = $\perp 010 \perp 1011 \perp$

$S_\perp$ = $\{1, 5, 10\}$ $S_{0,1}$ = $[10] \setminus S_\perp$

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$f^{(1)}(r')$:

Constants: $r_{S_{0,1}}$, $\cancel{r_{S_\perp}}$, $S_\perp$.

- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta\left(r'_{S_{0,1}}, r_{S_{0,1}}\right) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

Using injective length-tripling SCPRGs

Eve sees:

r_E = ⊥010⊥1011⊥

$S_\perp = \{1, 5, 10\}$ $\quad S_{0,1} = [10] \setminus S_\perp$

Eve does not know:

r = 1010010110

Replace with $SCPRG_\varepsilon(r_{S_\perp})$ for some choice of ε dependent on degradation condition. Here, $\varepsilon = \frac{1}{12}$.

$f^{(1)}(r')$:

Constants: $r_{S_{0,1}}, \cancel{r_{S_\perp}}, S_\perp$.

- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta(r'_{S_{0,1}}, r_{S_{0,1}}) < 0.1n + n^{0.9}$ output m
- Output ⊥ otherwise.

Using injective length-tripling SCPRGs

Eve sees:

$r_E = \perp 010 \perp 1011 \perp$

$S_\perp = \{1, 5, 10\}$

Eve does not know:

$r = 1010010110$

Parameter ε , dependent on degradation condition, is set so that Eve is unable to recover. Here, $\varepsilon = \frac{1}{12}$.

$f^{(2)}(r')$:

Constants: $r_{S_{0,1}}, SCPRG_\varepsilon(r_{S_\perp}), S_\perp$.

- Let $\alpha := SCPRG_\varepsilon.Recover\left(SCPRG_\varepsilon(r_{S_\perp}), r'_{S_\perp}\right)$.
- If $SCPRG_\varepsilon(\alpha) \neq SCPRG_\varepsilon(r_{S_\perp})$, then output $\perp$.
- Otherwise, set $r_{S_\perp} \leftarrow \alpha$.
- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta\left(r'_{S_{0,1}}, r_{S_{0,1}}\right) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

Using injective length-tripling SCPRGs

Eve sees:

r_E = $\perp 010 \perp 1011 \perp$

$S_\perp$ = $\{1, 5, 10\}$ $\qquad$ $S_{0,1}$ = $[10] \setminus S_\perp$

Eve does not know:

r = 1010010110

From Eve's point of view, $r_{S_\perp}$ is an unknown uniform random string.

$f^{(2)}(r')$:

Constants: $r_{S_{0,1}}, SC-PRG_\varepsilon(r_{S_\perp}), S_\perp$.

- Let $\alpha \coloneqq SCPRG_\varepsilon.Recover\left(SCPRG_\varepsilon(r_{S_\perp}), r'_{S_\perp}\right)$.
- If $SCPRG_\varepsilon(\alpha) \neq SCPRG_\varepsilon(r_{S_\perp})$, then output $\perp$.
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Using injective length-tripling SCPRGs

Eve sees:

$r_E = \perp 010 \perp 1011 \perp$

$S_\perp = \{1, 5, 10\}$ $S_{0,1} = [10] \setminus S_\perp$

Eve does not know:

$r = 1010010110$

Can therefore apply pseudorandomness property.

$f^{(3)}(r')$:

Constants: $r_{S_{0,1}}, R, S_\perp$.

- Let $\alpha \coloneqq SCPRG_\varepsilon.Recover(R, r'_{S_\perp})$.
- If $SCPRG_\varepsilon(\alpha) \neq R$, then output $\perp$.
- Otherwise, set $r_{S_\perp} \leftarrow \alpha$.
- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta(r'_{S_{0,1}}, r_{S_{0,1}}) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

Using injective length-tripling SCPRGs

Eve sees:

$r_E = \perp 010 \perp 1011 \perp$

$S_\perp = \{1, 5, 10\}$

Eve does not know:

$r = 1010010110$

With overwhelming probability R is not in the range of the $SCPRG$, so will be functionally equivalent to null circuit.

$f^{(3)}(r')$:

Constants: $r_{S_{0,1}}$, R, $S_\perp$.

- Let $\alpha \coloneqq SCPRG_\varepsilon.Recover(R, r'_{S_\perp})$.
- If $SCPRG_\varepsilon(\alpha) \neq R$, then output $\perp$.
- Otherwise, set $r_{S_\perp} \leftarrow \alpha$.
- If $\Delta(r'_{S_\perp}, r_{S_\perp}) + \Delta(r'_{S_{0,1}}, r_{S_{0,1}}) < 0.1n + n^{0.9}$ output m
- Output $\perp$ otherwise.

End of the Security Proof: Null Circuit

$f^{(4)}(r')$:

- Output $\perp$.

“Code Offset” construction of SCPRG

Injective PRG G.

List-decodable error correcting code $\mathcal{C}$ for up to $\frac{1}{2} - \varepsilon$ error rate for any constant $\varepsilon > 0$.

Concatenated code of binary Reed-Solomon codes with Hadamard code [Sudan, Trevisan, Vadhan ’99, Sudan ‘00]

$SCPRG_{\varepsilon}(s_1, s_2)$:

- Output $(s_1 + \mathcal{C}(s_2), G(s_2))$.

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$SCPRG_{\varepsilon}(s_1, s_2)$:

- Output $(s_1 + \mathcal{C}(s_2), G(s_2))$.

Pseudorandomness: s_1 is uniform random, so $s_1 + \mathcal{C}(s_2)$ is uniform random. Then, apply pseudorandomness of $G(s_2)$.

“Code Offset” construction of SCPRG

Injective PRG G.

List-decodable error correcting code $\mathcal{C}$ for up to $\frac{1}{2} - \varepsilon$ error rate for any constant $\varepsilon > 0$.

e.g. concatenated code of binary Reed-Solomon codes with Hadamard code
[Sudan, Trevisan, Vadhan ’99, Sudan ‘00]

$SCPRG_\varepsilon(s_1, s_2)$:

- Output $(s_1 + \mathcal{C}(s_2), G(s_2))$.

Self-correction: Can show, if $s_1', s'_2 \approx s_1, s_2$ and for appropriate lengths of s_1 and s_2, then $s_1' \approx s_1$.

Therefore, if $s_1', s'_2 \approx s_1, s_2$ then can recover a polynomial size list containing s_2 from $s_1 + \mathcal{C}(s_2)$.

Use $G(s_2)$ iterate over list to find s_2, then recover s_1.

Recap

We sketched the construction and security proof for a computational wiretap coding scheme for the non-degraded (BSC, BEC) case via iO & injective PRG.

Theorem: Assuming the existence of indistinguishability obfuscation (iO) and injective PRGs, there exists a computational wiretap coding scheme for any pair of non-degraded **binary-input** channels (ChB, ChE).

1. The given construction idea easily extends to the non-degraded $(BAC, BAEC)$ setting.

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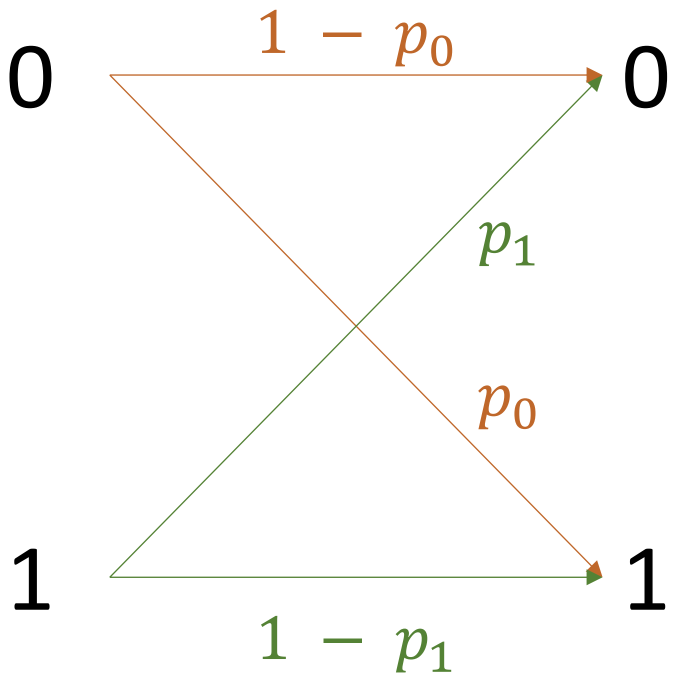
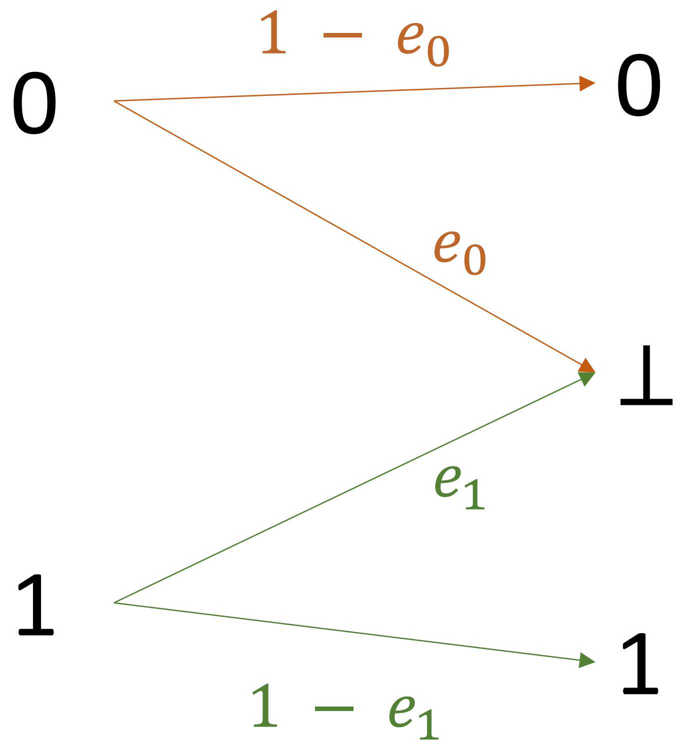


$$\begin{bmatrix} 1 - p_0 & p_0 \\ p_1 & 1 - p_1 \end{bmatrix}$$

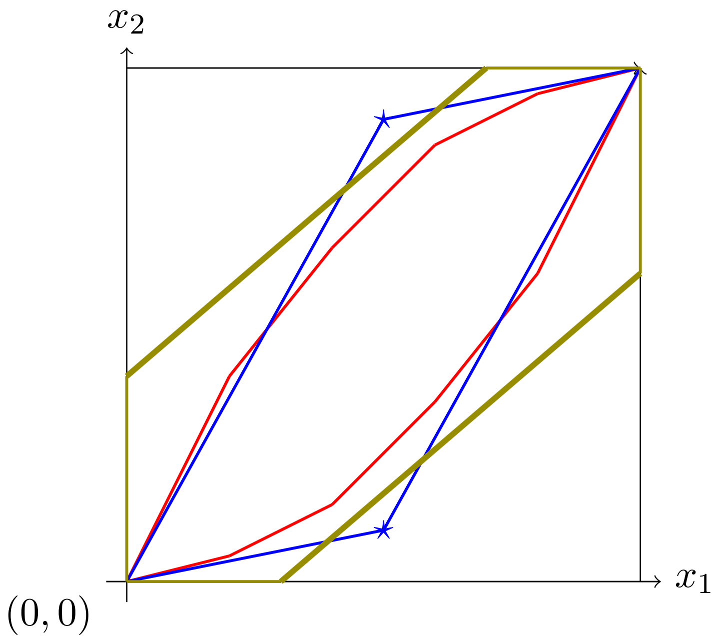


$$\begin{bmatrix} 1 - e_0 & 0 & e_0 \\ 0 & 1 - e_1 & e_1 \end{bmatrix}$$

Theorem: Assuming the existence of indistinguishability obfuscation (iO) and injective PRGs, there exists a computational wiretap coding scheme for any pair of non-degraded **binary-input** channels (ChB, ChE).

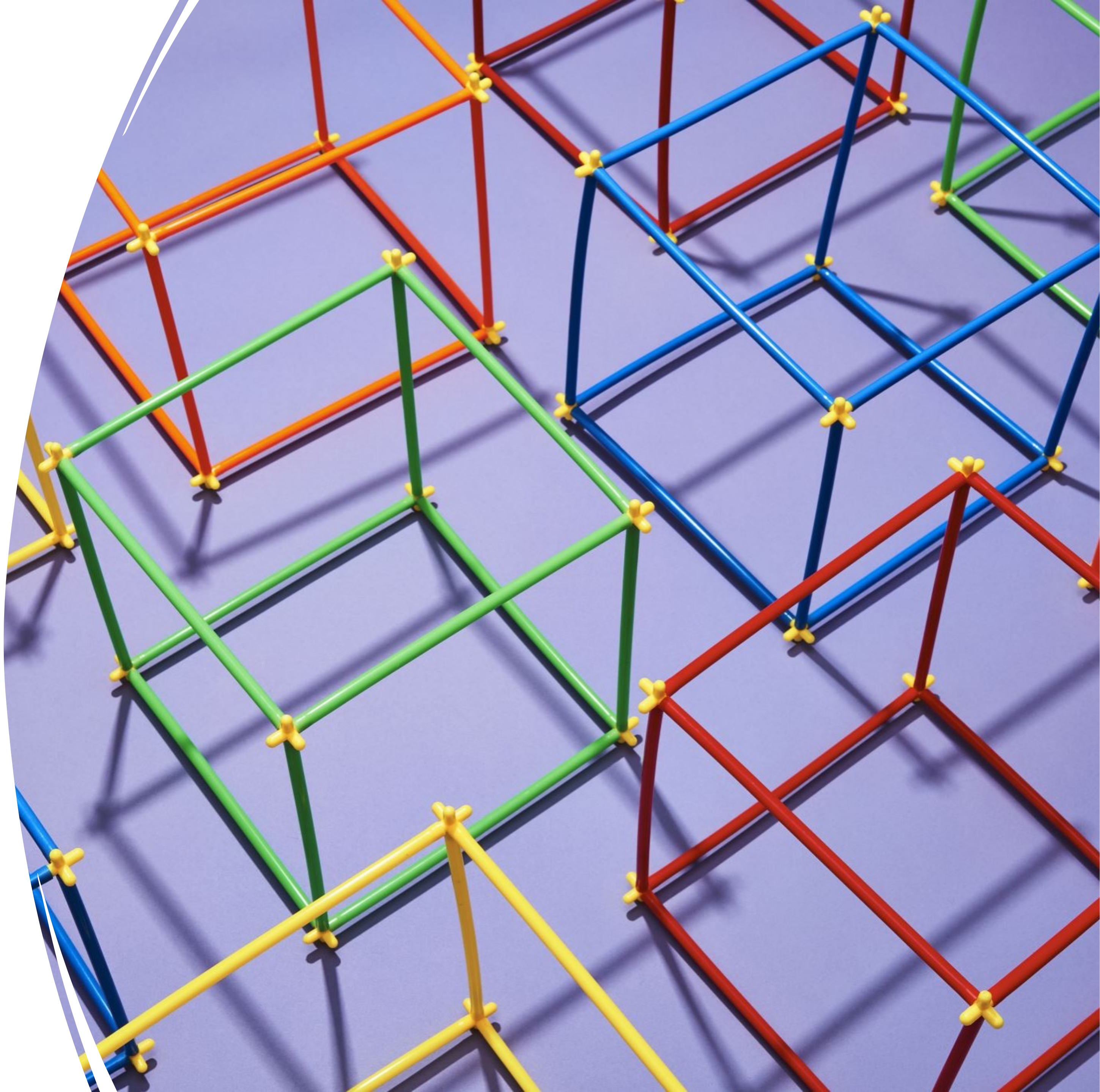
1. The given construction idea easily extends to the non-degraded $(BAC, BAEC)$ setting.

2. The case of every non-degraded binary-input channel pair (ChB, ChE) reduces to (1).



Some Open Directions

- Expanding construction beyond binary-input channels.
  - Characterize degradation for dimension three and beyond.
- Realizing computational wiretap coding from simpler cryptographic primitives or directly from hardness assumptions like LWE.
- Addressing the asterisk* in the initial riddle: Can we derandomize the encoding?

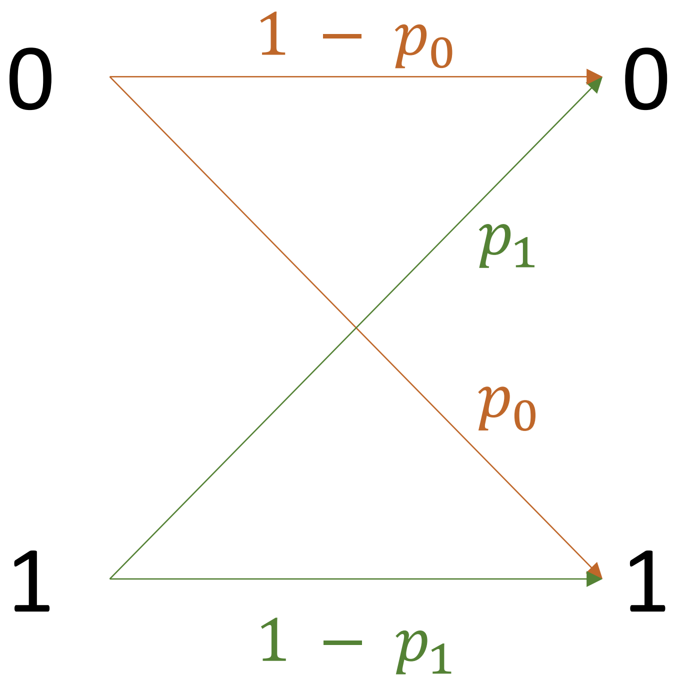


Thank you !

Appendix: The BAC/BAEC Case and General Binary-Input Case

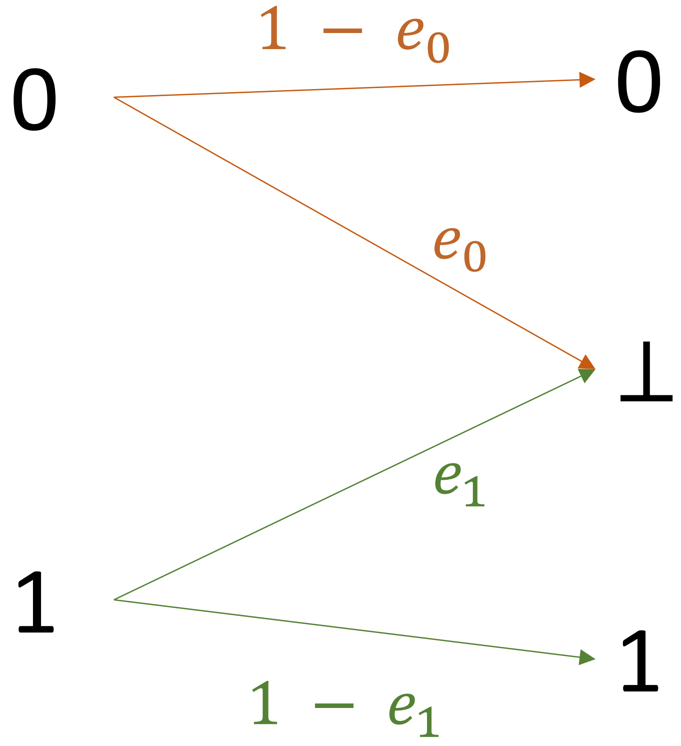
Asymmetric Binary Channels

Binary Asymmetric Channel (BAC)



$$\begin{bmatrix} 1-p_0 & p_0 \\ p_1 & 1-p_1 \end{bmatrix}$$

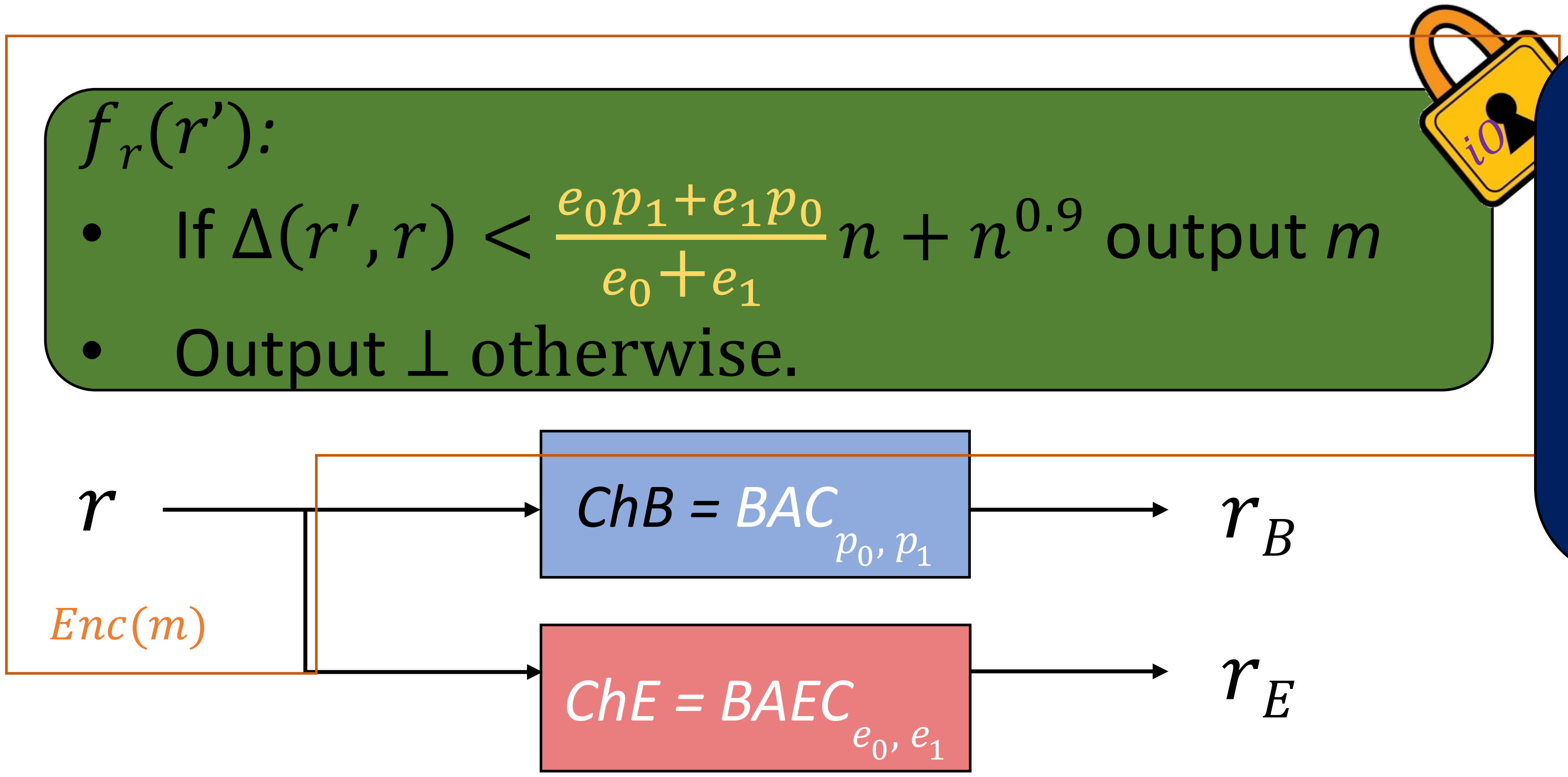
Binary Asymmetric Erasure Channel (BAEC)



$$\begin{bmatrix} 1-e_0 & 0 & e_0 \\ 0 & 1-e_1 & e_1 \end{bmatrix}$$

$$ChB = BAC_{p_0,\, p_1},\quad ChE = BAEC_{e_0,\, e_1}$$

Construction: Same as before, except initial distribution is such that from Eve's view, each erasure equally likely to have been 0 or 1.



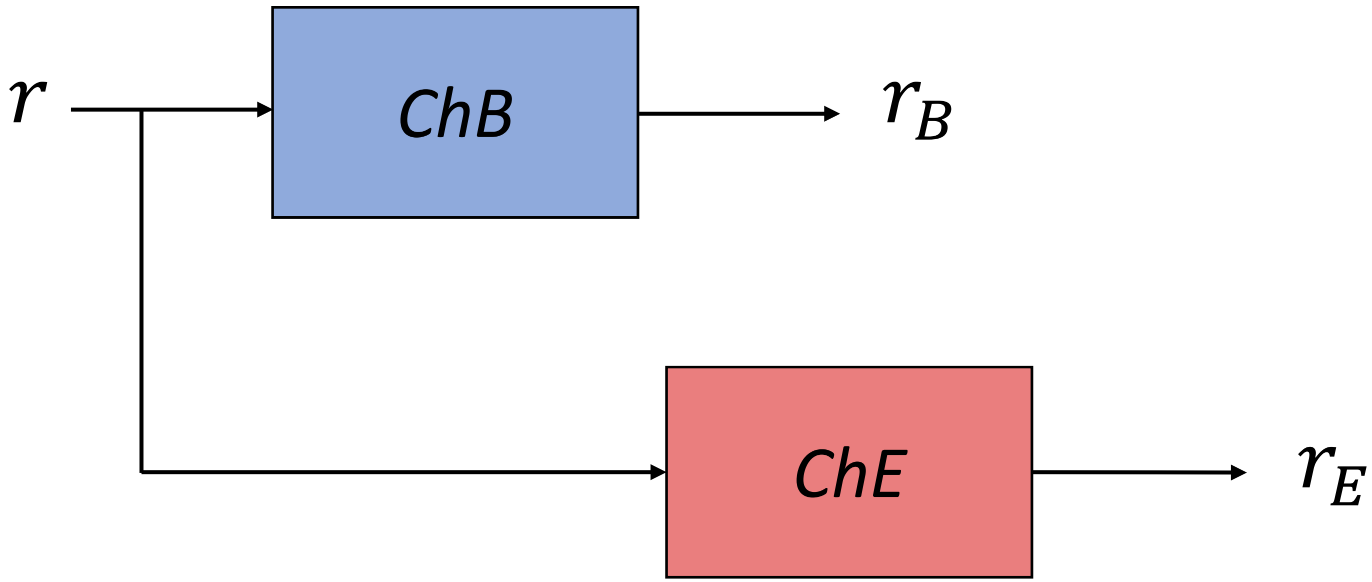
Then proceed by similar hybrid arguments as before.

Turns out, non-degradation condition for this channel pair
$e_0 p_1 + e_1 p_0 < e_0 e_1$
precisely implies Bob has enough advantage to recover m.

Pairs of Binary-input Channels Reduce to the BAC/BAEC Case

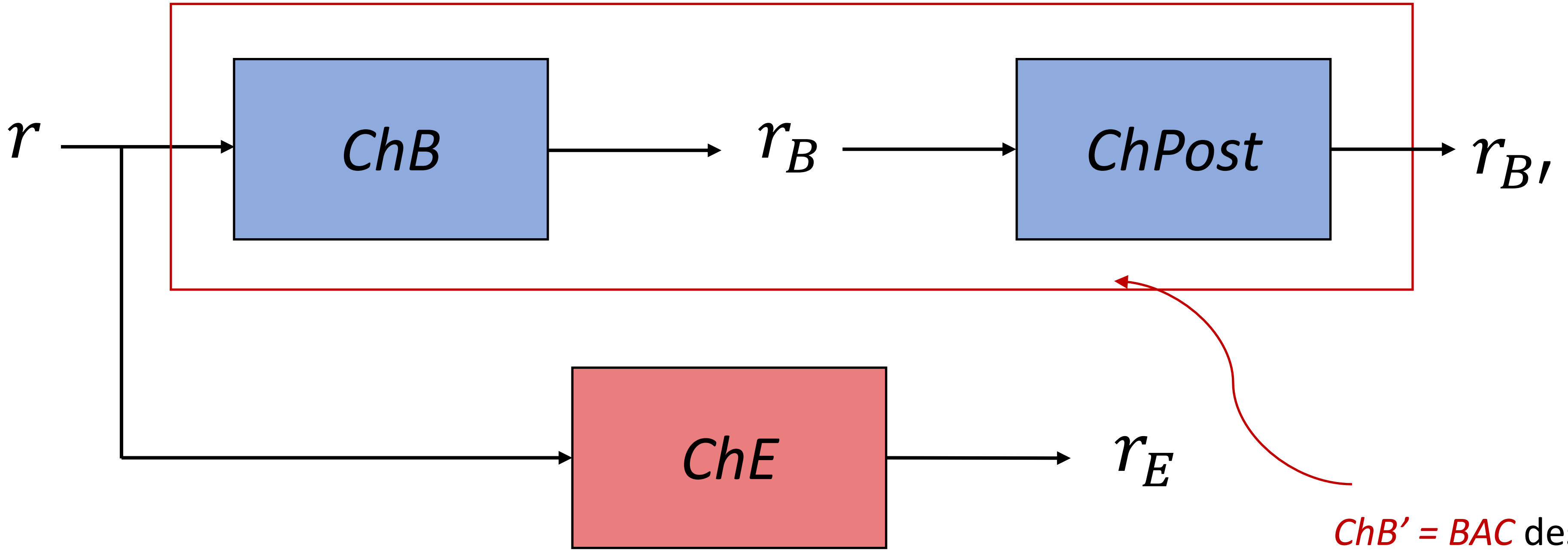
Pair of Arbitrary Binary Input Channels

Consider $\left(B = \begin{bmatrix} u_{11} & \cdots & u_{1n_B} \\ u_{21} & \cdots & u_{2n_B} \end{bmatrix}, E = \begin{bmatrix} u_{11} & \cdots & u_{1n_E} \\ u_{21} & \cdots & u_{2n_E} \end{bmatrix}\right)$ s.t. B not a degradation of E.



Reducing Pair of Arbitrary Binary Input Channels to BAC/BAEC Case: Bob's Output Alphabet

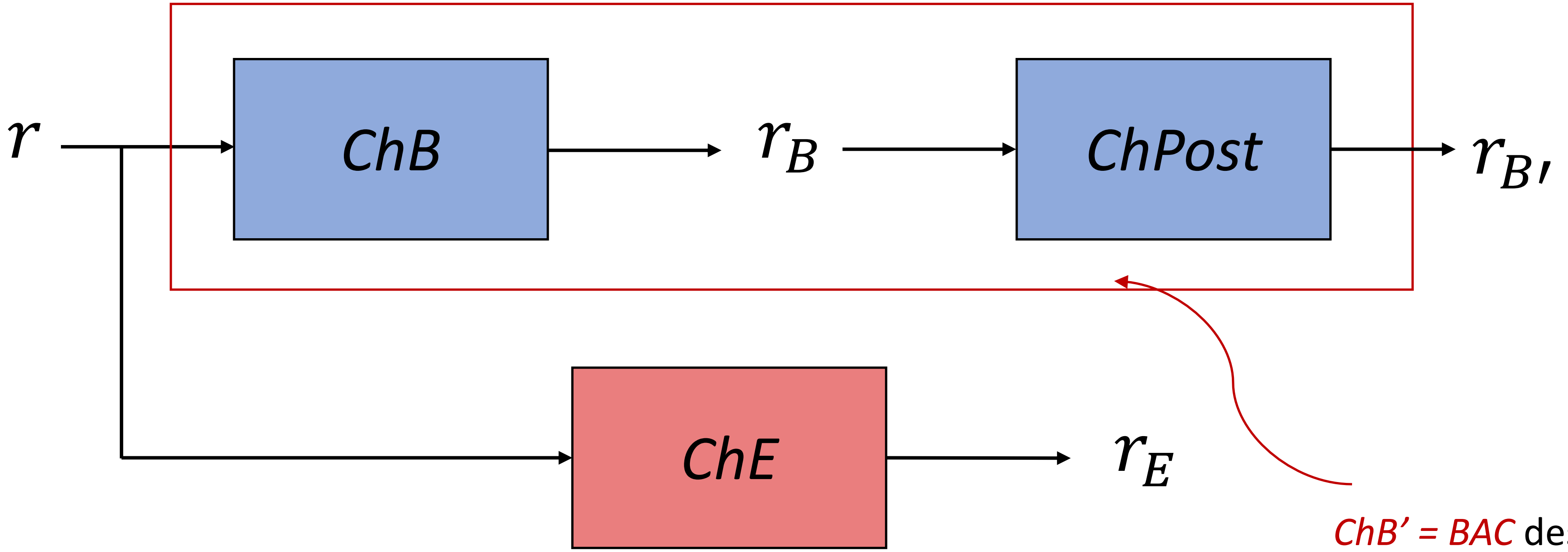
Consider $\left(B = \begin{bmatrix} u_{11} & \cdots & u_{1n_B} \\ u_{21} & \cdots & u_{2n_B} \end{bmatrix}, E = \begin{bmatrix} u_{11} & \cdots & u_{1n_E} \\ u_{21} & \cdots & u_{2n_E} \end{bmatrix}\right)$ s.t. B not a degradation of E.



ChB' = BAC described by some matrix B'

Reducing Pair of Arbitrary Binary Input Channels to BAC/BAEC Case: Bob's Output Alphabet

Consider ($B' = \begin{bmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{bmatrix}$, $E = \begin{bmatrix} v_{11} & \cdots & v_{1n_E} \\ v_{21} & \cdots & v_{2n_E} \end{bmatrix}$) s.t. B not a degradation of E.



ChB' = BAC described by some matrix B'

Find B' s.t. (1) B' not a degradation of E.
(2) B' degradation of B.

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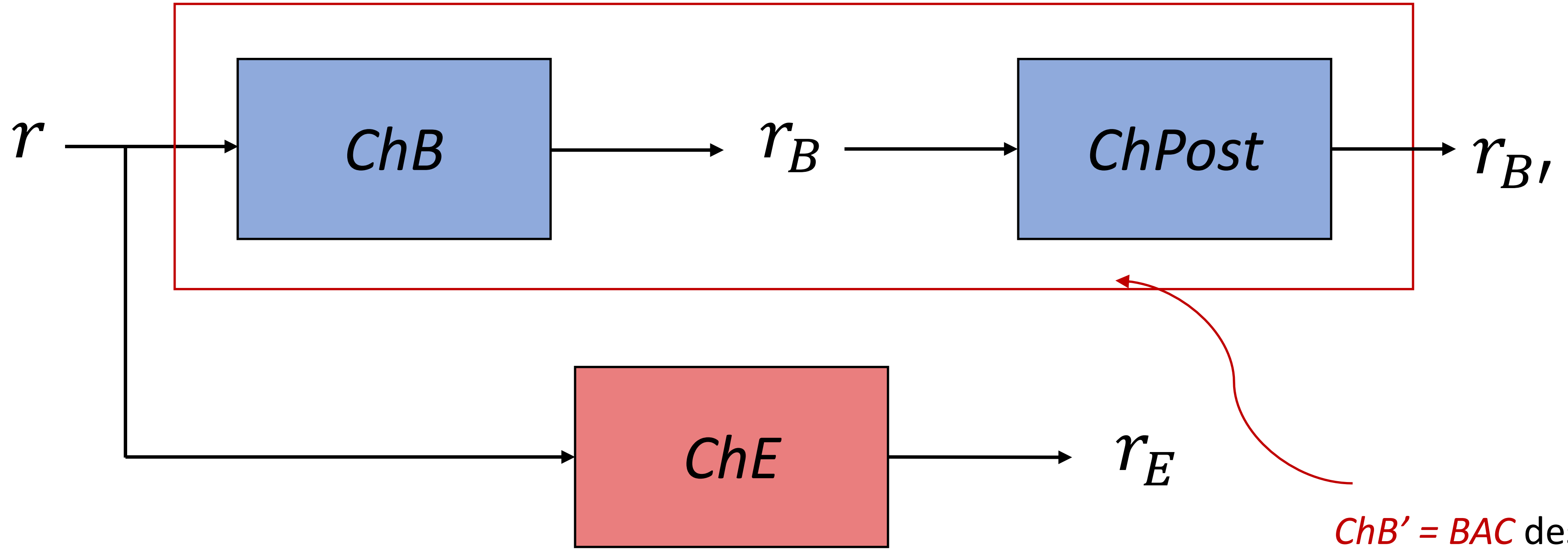


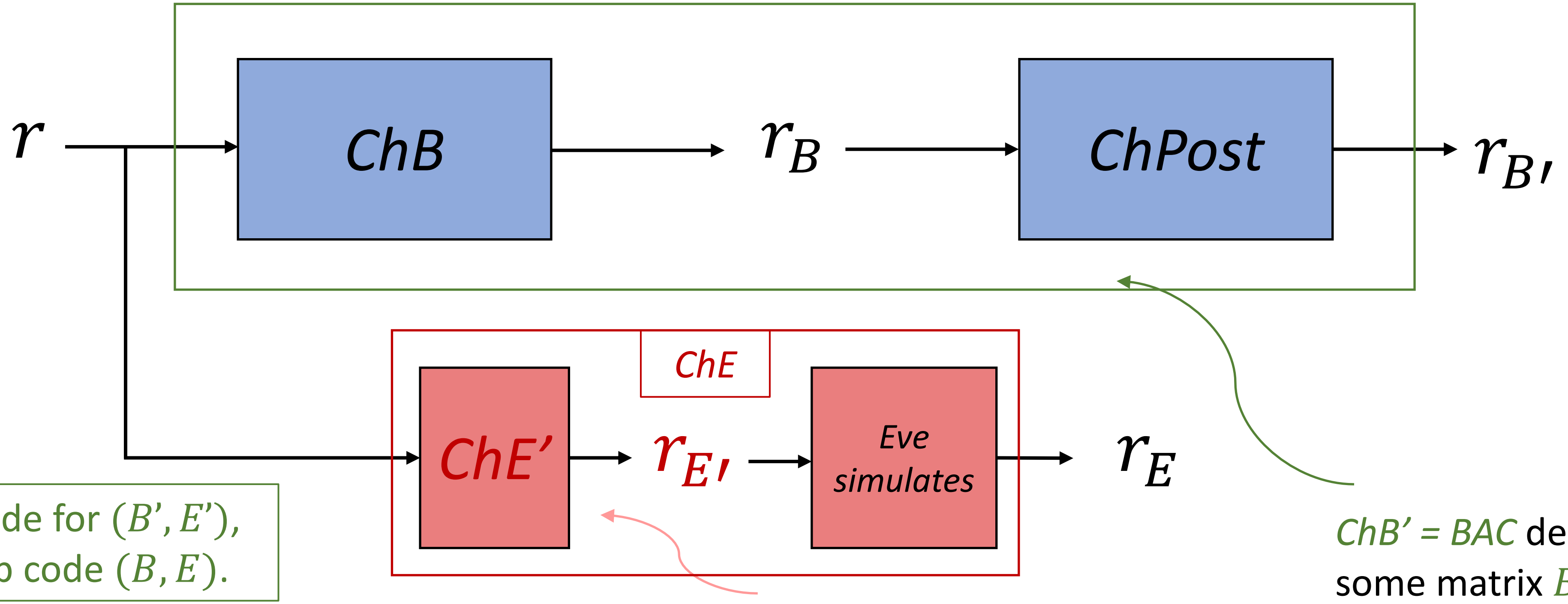
ChB' = BAC described by some matrix B'

Find B' s.t. (1) B' not a degradation of E.
(2) B' degradation of B.

Any wiretap code for (B', E), gives a wiretap code (B, E).

Reducing Pair of Arbitrary Binary Input Channels to BAC/BAEC Case: Simulating ChE with a BAEC

Consider $\left(B' = \begin{bmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{bmatrix}, E = \begin{bmatrix} v_{11} & \cdots & v_{1n_E} \\ v_{21} & \cdots & v_{2n_E} \end{bmatrix}\right)$ such that $\mathcal{P}(B') \nsubseteq \mathcal{P}(E)$, $\mathcal{P}(B') \subseteq \mathcal{P}(B)$.



Any wiretap code for (B', E'), gives a wiretap code (B, E).

ChB' = BAC described by some matrix B'

Imagine that Eve instead receives an output through *ChE' = BAEC* described by some matrix E', effectively giving Eve even more information, but hopefully not enough to simulate B'!

Reducing Pair of Arbitrary Binary Input Channels to BAC/BAEC Case: Simulating ChE with a BAEC

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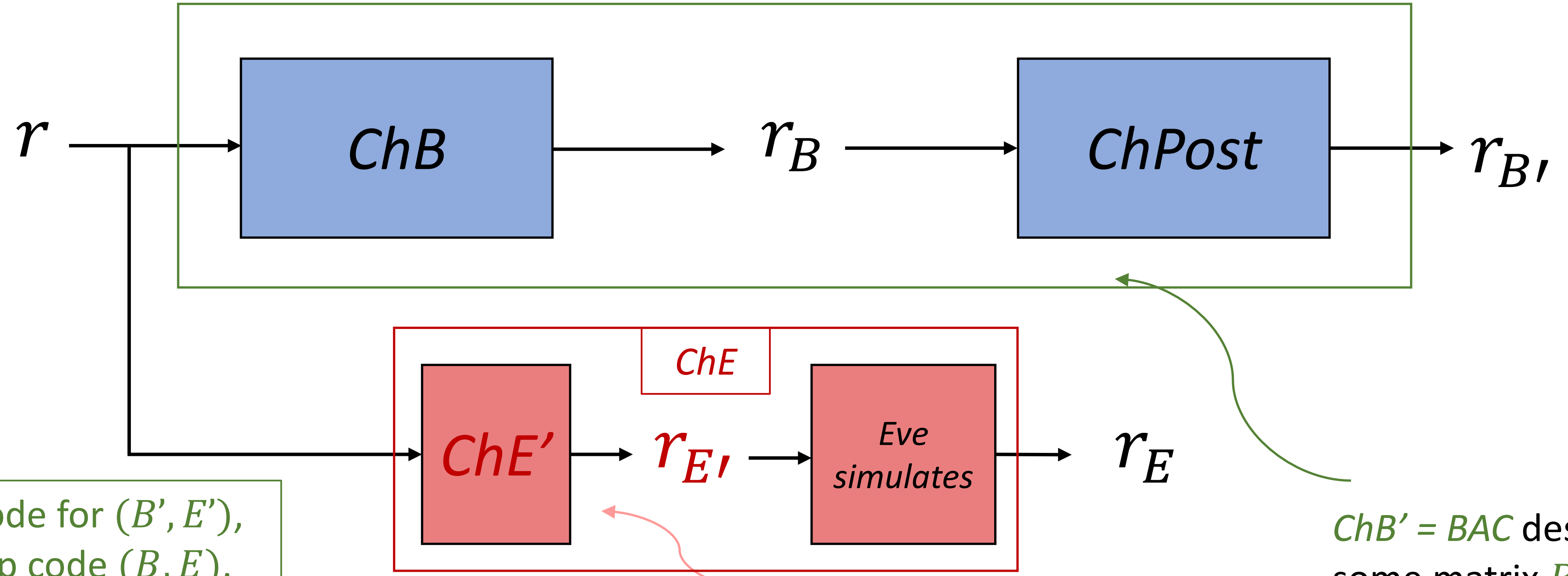

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ChB' = BAC described by some matrix B'

Imagine that Eve instead receives an output through *ChE' = BAEC* described by some matrix E', effectively giving Eve even more information, but hopefully not enough to simulate B'!

Finding BAEC E' via Polytope Formulation

A New Polytope formulation

Def: [Channel Polytope] Let A be a matrix of non-negative entries. We associate to A the following polytope, denoted $\mathcal{P}(A)$, which can be defined in either of the following equivalent ways:

- $\mathcal{P}(A)$, is the convex hull of all subset-sums of columns of A.
- $\mathcal{P}(A) = \{Av : 0 \leq v \leq 1\}$.

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Theorem: Let $B \in \mathbb{R}^{2\times n_B}$ and E $\in \mathbb{R}^{2\times n_E}$ be arbitrary row-stochastic matrices. Then, $B \neq E \cdot S$ for every row stochastic matrix S if and only if $\mathcal{P}(B) \not\subseteq \mathcal{P}(E)$.

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- $\mathcal{P}(A) = \{Ax : 0 \leq x \leq 1\}$

In the interest of time, we will not sketch the proof.

If row count > 2, then this is false. Explicit counterexample for case of 3.

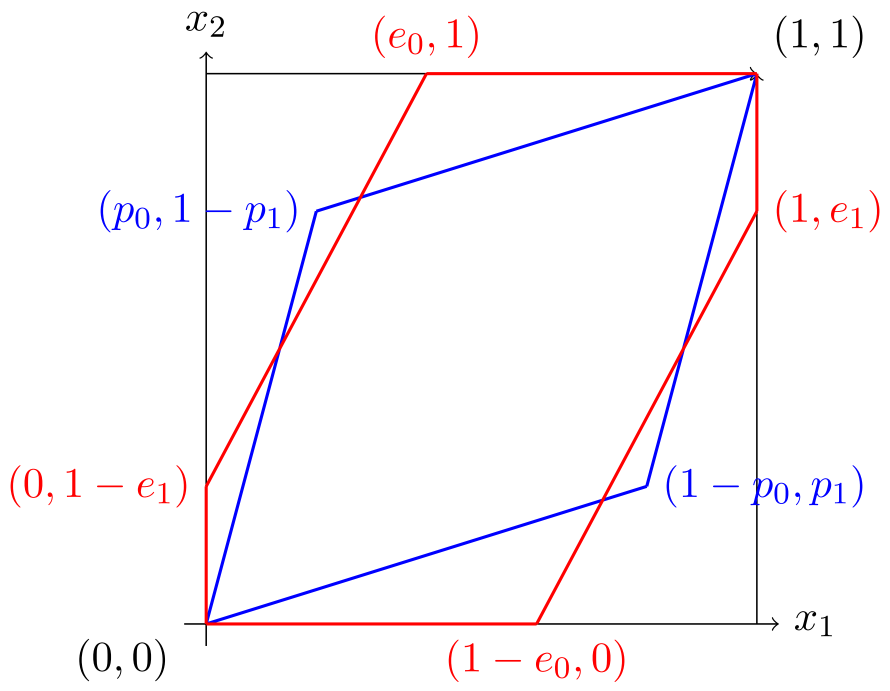
Theorem: Let $B \in \mathbb{R}^{2 \times n_B}$ and $E \in \mathbb{R}^{2 \times n_E}$ be arbitrary row-stochastic matrices. Then, $\mathrm{Ch}B$ is not a degradation of $\mathrm{Ch}E$ if and only if $\mathcal{P}(B) \nsubseteq \mathcal{P}(E)$.

Polytope Example

Binary Asymmetric Erasure Channel (BAEC)

$$\begin{bmatrix} 1-p_0 & p_0 \\ p_1 & 1-p_1 \end{bmatrix} \qquad \begin{bmatrix} 1-e_0 & 0 & e_0 \\ 0 & 1-e_1 & e_1 \end{bmatrix}$$

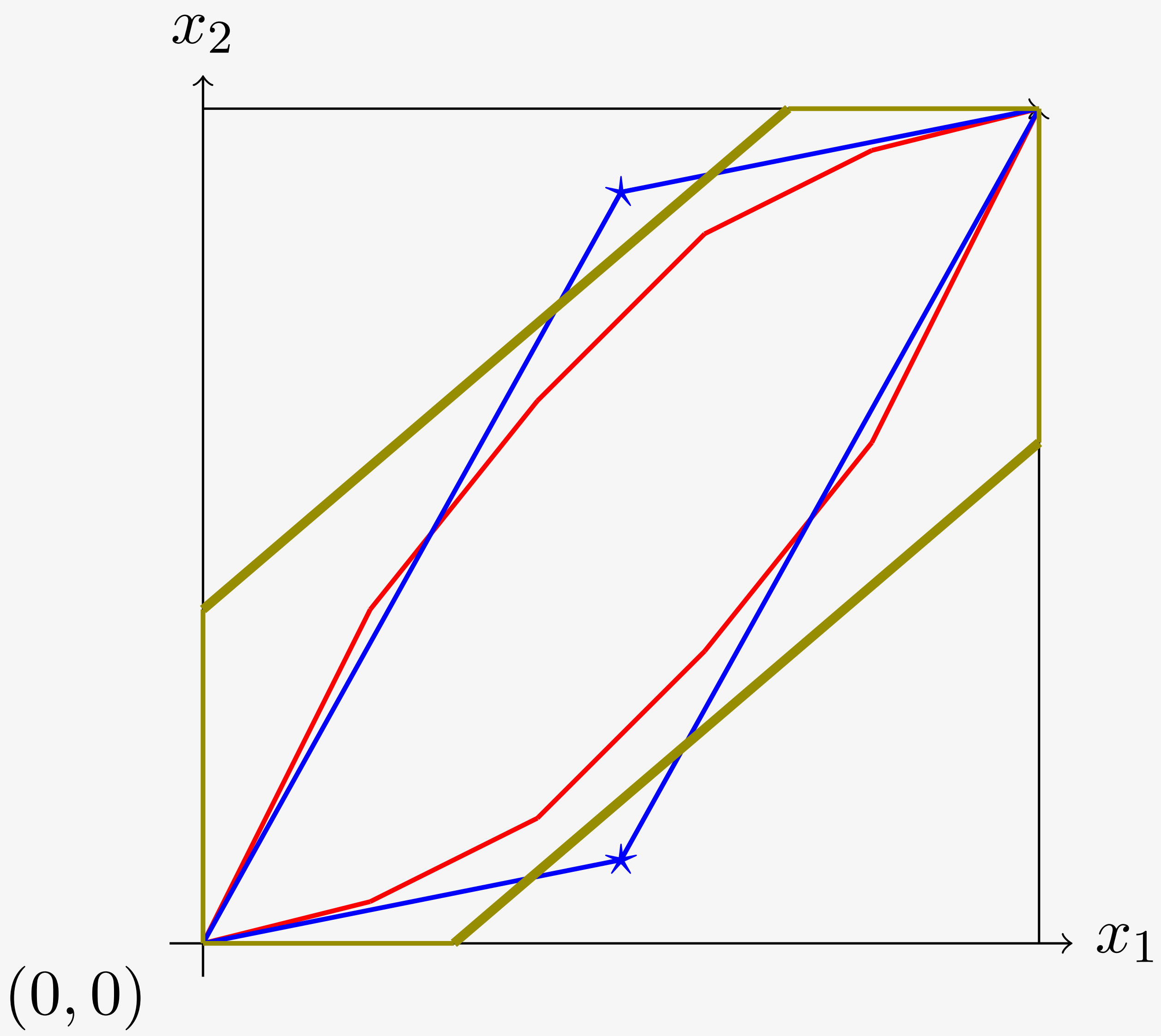
Binary Asymmetric Channel (BAC)



The blue polytope corresponds to the BAC.

The red polytope corresponds to the BAEC.

Since the blue polytope is **not** contained in the red polytope, the BAC channel is **not** a degradation of the BAEC channel.

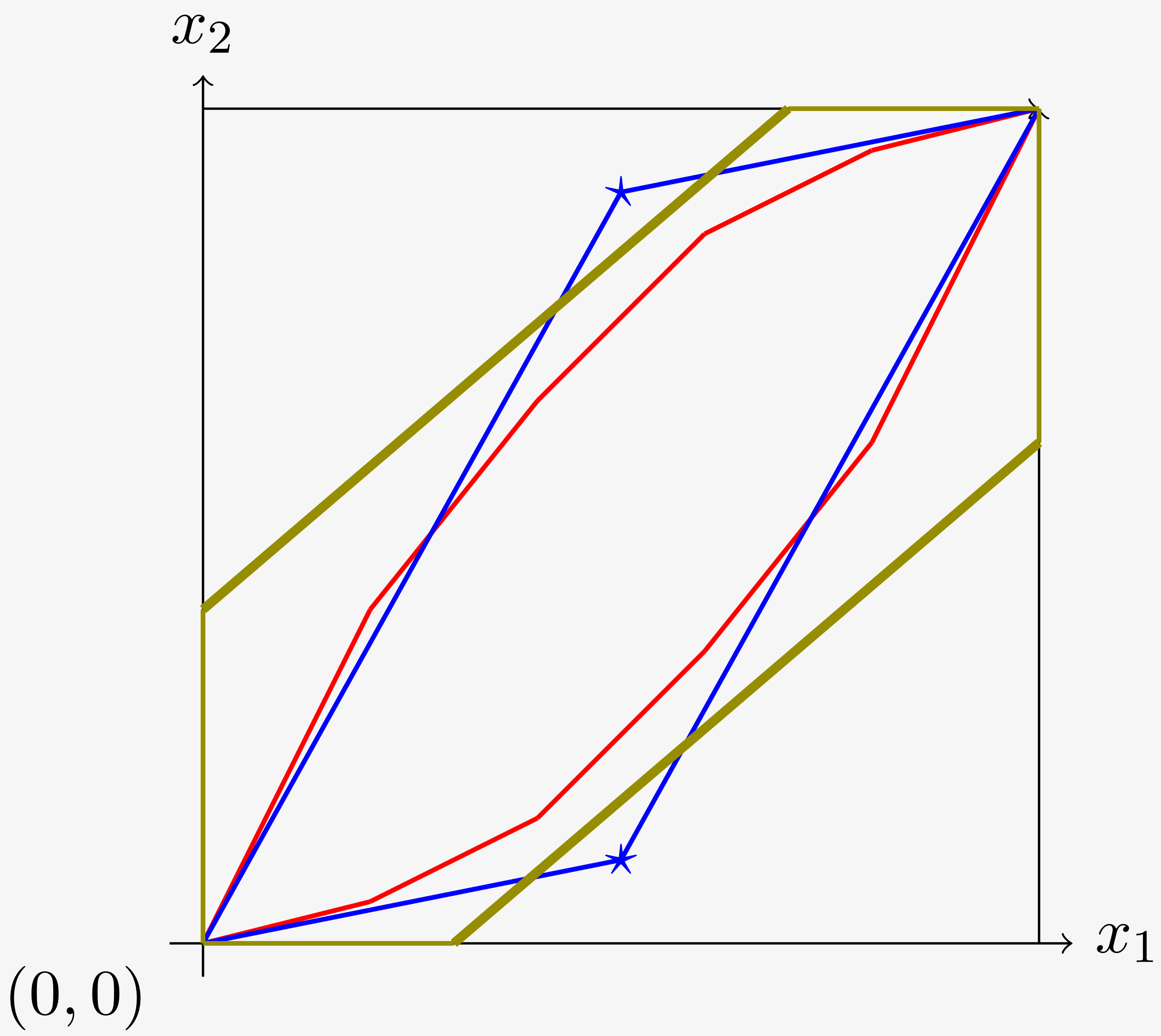


Reducing Eve's Channel to a BAEC

The blue polytope corresponds to the BAC.

The red polytope corresponds to some channel ChE.

Since the blue polytope is **not** contained in the red polytope, the BAC channel is **not** a degradation of ChE.



Reducing Eve's Channel to a BAEC

Apply the strict separating hyperplane theorem!

Take an extreme point of the BAC **not** inside the ChE polytope and separate it from the ChE polytope.

Olive polytope is a BAEC channel s.t. (1) ChE is a degradation and (2) ChB is not a degradation.

Can find this polytope efficiently.