Relinearization attack on LPN over $\mathbb{F}_p$
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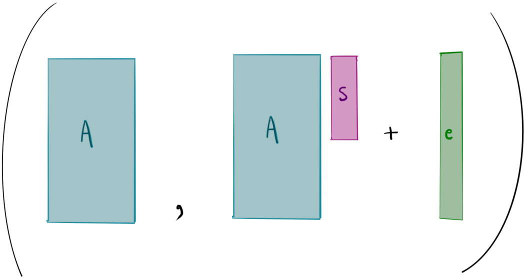
Why do we care about attacking LPN over large fields?

- LPN over large fields [IPS09] is an important assumption in current indistinguishability obfuscation constructions [JLS21].
- Important to understand its security: so far a naive sub-exponential guessing algorithm is still the state-of-the-art.

Will Gröbner Bases Work?

Does a linearization/Gröbner bases attack work? So far, nope :(

LPN over $\mathbb{F}_p$



- $\mathbf{A} \leftarrow \mathbb{F}_p^{m \times n}$, $\mathbf{s} \leftarrow \mathbb{F}_p^n$ where p is a λ-bit prime (sec. param λ).
- For sparsity constant γ, for $i \in [m]$, $\mathbf{e}_i \leftarrow \begin{cases} \mathbb{F}_p & \text{with prob. } n^{-\gamma} \\ 0 & \text{otherwise} \end{cases}$
- Number of equations $m = n^{1+\alpha}$ (Think constant $\alpha < 1$).

Goal: Recover $\mathbf{s}$ from $(\mathbf{A}, \mathbf{As} + \mathbf{e})$ (unique $\mathbf{s}$ w.h.p.)

Some use cases of LPN over $\mathbb{F}_p$ in cryptography

Using the decisional variant (there's a search-to-decision reduction):

- Public-key encryption [Ale03; DP12; AAB15] (when sparsity $\gamma \geq 1/2$)
- Vector oblivious linear-function evaluation (VOLE) generators [Boy+18]
- Indistinguishability obfuscation [JLS21]

Known attack landscape for search LPN over $\mathbb{F}_p$

- No known reductions between LPN over $\mathbb{F}_p$ and LWE (different error distributions).
- **Folklore attack** (low noise rate $n^{-\gamma}$): repeatedly take n samples, assume error-free, and solve for **s** via Gaussian elimination [Car+09; EKM17].
  - Expected runtime: $1/(1-n^{-\gamma})^n$.
    - If $\gamma \geq \frac{1}{2}$, this is $O\left(e^{n^{1-\gamma}}\right)$.
    - If $\gamma < 1/2$, then it's $e^{O(n^{1-\gamma})}$.
  - Sample complexity: $O(n^{1+\gamma})$.
- Information set decoding and variants [Pra62; CS16].
- For high noise rate (e.g. constant): BKW algorithm with runtime, memory, and sample complexity $O\left(2^{n/\log n}\right)$ [BKW]. Scaled-down version works with polynomial sample size but worse runtime [Lyu05].
- What about Gröbner basis attacks?

Our objective and contributions

- **Our regime**: low noise rate $n^{-\gamma}$ and sample complexity $m = n^{1+\alpha}$ for $\alpha \in (0, 1)$.
  - No known attack better than the folklore attack.
- **Objective**: Find a better subexponential attack via a Gröbner basis approach.

We didn't succeed.

- Our approach only yields an exponential time attack, assuming a widely believed conjecture about "semi-regularity".
- We discuss the approaches we tried and some open questions.

What is linearization?

Linearization technique [KS99; AG11]: replace all the monomials with a new set of variables to obtain a linear system

$$x_1 \mapsto y_1$$
$$x_1 x_2 \mapsto y_{1,2}$$
$$x_1 x_2 + x_1 + 3 \mapsto y_{1,2} + y_1 + 3$$

- Starting with m degree-d equations, the number of monomials present is the number of new variables. At most $n' = \binom{n+d}{d}$ many.
- If initially there was a unique solution and the number of equations m is sufficiently larger than n', then the linearized system has the same unique solution with high probability.
- Solving the resulting polynomial system takes time approximately $O\left((n')^{\omega}\right)$ for linear algebra constant $2 \leq \omega \leq 3$.

An example: linearization attack on Binary LWE

Binary LWE setting: each error $e_i \in \{0, 1\}$ where $e_i \sim \text{Ber}(\tau)$. Given $(\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e})$, recover $\mathbf{s}$.

Due to Arora-Ge [AG11]:

- Since the errors are in $\{0, 1\}$, we have m degree 2 equations in $s_1, \ldots, s_n$:

$$\left\{(b_i - \mathbf{a}_i \cdot \mathbf{s} - 1) \cdot (b_i - \mathbf{a}_i \cdot \mathbf{s}) = 0 \bmod p\right\}_{i \in [m]}$$

- Linearize $s_i \mapsto y_i$, $s_i s_j \mapsto y_{i,j}$. Number of linearized variables is $O(n^2)$.
- This gives polynomial time recovery if $m = \Omega(n^2)$.
- Sample-time tradeoff for samples $m = n^{1+\alpha}$ characterized by Sun et al. [STA20].

Smaller sample complexity $m \sim n^{1+\alpha}$ (1/2)

- **Issue**: When $m \sim n^{1+\alpha}$ for $\alpha \in (0, 1)$, there are not enough equations for the linearized system to have a unique solution.
- **Goal**: Generate more equations.

Smaller sample complexity $m \sim n^{1+\alpha}$ (2/2)

Degree-d **Macaulay expansion**: multiply every equation by all monomials up to degree d (can view as a matrix of coefficients, the Macaulay matrix):

$$
\begin{aligned}
&\left\{ f_i(x_1, \ldots, x_n) \right\}_{i \in [m]} \\
&\quad \cup \left\{ x_1 \cdot f_i(x_1, \ldots, x_n) \right\}_{i \in [m]} \\
&\quad \cup \left\{ x_2 \cdot f_i(x_1, \ldots, x_n) \right\}_{i \in [m]} \\
&\quad \cup \left\{ x_1 x_2 \cdot f_i(x_1, \ldots, x_n) \right\}_{i \in [m]} \\
&\quad \vdots
\end{aligned}
$$

Macaulay expansion finds our unique solution

- **Intuition**: if there is a unique solution to $\{f_1(\mathbf{x}) = 0, \ldots, f_m(\mathbf{x}) = 0\}$, say $\mathbf{s}$, then Hilbert's Nullstellensatz says ideal

$$\langle f_1, \ldots, f_m \rangle$$

is equivalent to the ideal (whose generators are our Gröbner basis)

$$\langle x_1 - s_1, \ldots, x_n - s_n \rangle.$$

Therefore, there exist some polynomials (WLOG of minimal degree) $\{g_{i,j}\}_{i \in [m], j \in [n]}$ such that for all $j \in [n]$

$$x_j - s_j = \sum_{i \in [m]} g_{i,j} \cdot f_i$$

Punchline: Expand until we can recover the Gröbner basis $(x_1 - s_1, \ldots, x_n - s_n)$.

Macaulay expansion intimately related with Gröbner bases

Computing a Gröbner basis for a homogeneous polynomial system $(f_1, \ldots, f_m)$ is equivalent to performing Gaussian elimination on Macaulay matrices [Laz83].

Recall LPN over $\mathbb{F}_p$

Recall the setup: Our input is $(\mathbf{A}, \mathbf{b})$ where

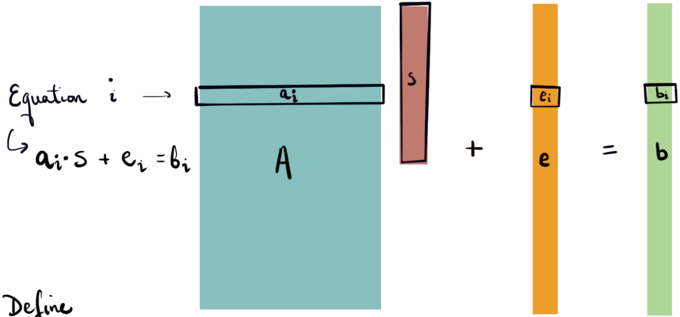
- $\mathbf{A} \leftarrow \mathbb{F}_p^{m \times n}$ where $m = n^{1+\alpha}$ samples, $\alpha \in (0, 1)$ constant.
- $\mathbf{b} = \mathbf{A} \cdot \mathbf{s} + \mathbf{e}$ where $\mathbf{s} \leftarrow \mathbb{F}_p^n$ and $\mathbf{e} = (e_1, \ldots, e_m)$ such that for constant sparsity parameter $\gamma \in (0, 1)$

$$\mathbf{e}_i \stackrel{\$}{\leftarrow} \begin{cases} \mathbb{F}_p & \text{with probability } n^{-\gamma} \\ 0 & \text{otherwise} \end{cases}$$

To solve for $\mathbf{s}$, we'll construct a quadratic system of equations.

Our approach: guess whether an equation has error

There's no bound on the error size, so instead we'll guess whether an equation has error:



Define

$$\alpha_i = \begin{cases} 1 & \text{if } e_i = 0 \\ 0 & \text{if } e_i \neq 0 \end{cases}$$

Gives equations:

$$\{ \alpha_i (a_i \cdot s) = \alpha_i b_i \}_{i=1}^{m}$$

Our system of equations for LPN over $\mathbb{F}_p$ (2/2)

- Variables:
  - $\mathbf{x} = (x_1, \ldots, x_n)$ for the secret.
  - $\alpha_1, \ldots, \alpha_m$ will be indicator variables for error-free equations so that $\alpha_i = 1$ if ith equation is error-free, 0 otherwise.
  - Number of initial variables is $N := n + m$.
- Equations:
  - Guess the number of error-ridden equations t where $t \in [m]$.

$$\mathcal{F} \triangleq \left\{ \alpha_i \mathbf{a}_i \cdot \mathbf{x} = \alpha_i b_i \right\}_{i \in [m]} \cup \left\{ \alpha_i(\alpha_i - 1) = 0 \right\}_{i \in [m]} \cup \left\{ t = m - \sum_{i \in [m]} \alpha_i \right\}$$

  - Number of initial equations is $2m + 1$.

Initial hopes for a subexponential attack

Initially, $N = n + m$ variables and $2m + 1$ equations.

After d-degree Macaulay expansion,

- The number of variables is at most the number of monomials of degree at most $d + 2$: $V_d = \binom{N+d+2}{d+2}$
- The number of equations is $E_d = (2m+1)\binom{N+d}{d}$.
- $E_d \geq V_d$ when $d = \Omega\left(\sqrt{m}\right)$.

Assuming full rank of the expanded system, we see that Gaussian elimination on $O\left(\sqrt{m}\right)$-degree expanded system takes time $O\left(\binom{n+m+\sqrt{m}}{\sqrt{m}}^{\omega}\right) = e^{O(\sqrt{m}\ln m)}$.

The issue of rank of the expanded Macaulay matrix

Is the full rank assumption with an $O(\sqrt{m})$ expansion justified?

What degree of Macaulay expansion do we actually need so that the linearized expanded system of polynomials has full rank?

“Semi-regularity” implies exponential time attack

Main Problem: If our initial polynomial system is “semi-regular”, then $O(m)$-degree expansion is necessary (the runtime therefore is exponential).

What is a semi-regular polynomial system?

For our purposes:

- Semi-regular polynomial systems are sequences for which we can estimate a runtime upper bound for computing the Gröbner basis. (i.e. via a characterization for the Hilbert polynomial w.r.t grevlex order)
- Random overdetermined ($m > n$) polynomial systems are conjectured to be semi-regular (related to Fröberg's conjecture (1985), an open algebraic-geometric question).

Estimating runtime with the degree of regularity

Assuming a polynomial system is semi-regular, characterizing the attack complexity reduces to computing the degree of semi-regularity.

Lemma ([BFS15; Alb+15])

Let $f_1, \ldots, f_m \in \mathbb{F}_p[x_1, \ldots, x_n]$ where $m > n$. If $(f_1, \ldots, f_m)$ semi-regular, then the number of field operation required to compute a Gröbner basis of the ideal $\langle f_1, \ldots, f_m \rangle$ for any graded monomial ordering is bounded by

$$O\left(m \cdot d_{reg} \binom{n + d_{reg} - 1}{d_{reg}}^{\omega}\right), \text{ as } d_{reg} \to \infty$$

where ω is the linear algebra constant and d_{reg} is the degree of regularity of $\langle f_1, \ldots, f_m \rangle$.

Semi-regularity for homogeneous polynomials

Definition ([Alb+15])

Let $m \geq n$, let $(f_1, \ldots, f_m) \in \mathbb{F}_p[x_1, \ldots, x_n]$ be homogeneous polynomials of degree $d_1, \ldots, d_m$ resp. and let $\mathcal{I}$ be the ideal generated by these polynomials. The system is said to be a semi-regular sequence if the Hilbert polynomial associated to $\mathcal{I}$ w.r.t. to the grevlex order is

$$H(z) = \left[\frac{\prod_{i=1}^{m}(1 - z^{d_i})}{(1 - z)^n}\right]_+$$

where $[S]_+$ is the polynomial obtained by truncating the series S before the index of its non-positive coefficient.

The degree of regularity of a semi-regular sequence is $1 + \deg(H(z))$.

Homogenization of arbitrary polynomials

Definition ([Alb+15])

Let $f_1, \ldots, f_m \in \mathbb{F}_p[x_1, \ldots, x_n]$ be arbitrary (possibly inhomogeneous) polynomials. Let $f_1^h, \ldots, f_m^h$ be their respective homogeneous components of highest degree. A sequence $(f_1, \ldots, f_m)$ is semi-regular if the sequence $(f_1^h, \ldots, f_m^h)$ is semi-regular.

- e.g. if $f = 1 + x_1 + x_1x_2 + x_1^2$, then $f^h = x_1x_2 + x_1^2$.

Assuming semi-regularity in our setting

- First, a simplification:

$$\alpha_1 = m - t - \sum_{i \neq 1} \alpha_i$$

eliminate the variable α_1 by substitution to obtain $E = 2m$ equations (all are degree 2), $V = n + m - 1$ variables.

- After the simplification, our Hilbert series assuming semi-regularity is

$$H_{E,V}(z) = \frac{(1 - z^2)^E}{(1 - z)^{V+1}} = \sum_{d=0}^{\infty} h_d z^d$$

- Degree of regularity, d_{reg} is the first d such that h_d is non-positive.

Computing the degree of regularity in our setting

Sun et al. [STA20] perform the same computation for a different polynomial system for Binary LWE:

- Saddle point approximation to estimate the behavior of the coefficients of the Hilbert series:

$$d_{reg} + 1 = E - \frac{V+1}{2} - \sqrt{E(E-V)}$$

Theorem

Consider an $\mathrm{LPN}(n, m, \gamma)$ *instance with* $m = n^{1+\alpha}$. *Assuming semi-regularity, the degree of regularity of our system* $\mathcal{F}$ *behaves asymptotically as*

$$d_{reg} \approx 0.09n^{1+\alpha} + 0.2n + 0.18n^{1-\alpha} + o(n^{-2\alpha}) = O(m)$$

Can we directly increase the rank of the Macaulay matrix?

Observation: α_i variables are indicators for *sparse* errors. The product $\alpha_{i_1} \cdots \alpha_{i_d} = 0$ with high probability for large d.

- How many of these equations can we add? Subexponentially many.

Theorem

Consider an $\mathrm{LPN}(n, m, \gamma)$ *instance with* $m = n^{1+\alpha}$*. We assume that the number of instances with errors is* $t = \frac{m}{n^\gamma}$*. Pick* $\delta \in (0, 1)$ *sufficiently small and* $d \in \mathbb{Z}^+$ *such that* $d = \lceil n^{\gamma+\gamma'} \rceil$ *where* $\gamma' < 1 + \alpha$*. Then we can introduce up to* $k = \left\lfloor -\ln(1-\delta) 2^{n^{\gamma'}} \right\rfloor$ *equations of the form* $\alpha_{i_1} \cdots \alpha_{i_d} = 0$ *where the* i_j *are distinct for each equation, and all* k *equations hold with probability* $1 - \delta$*.*

- We don't know how these equations affect the rank of the Macaulay matrix.

Difficulty of Estimating Rank

- Estimating the rank of even the standard Macaulay matrix is quite challenging. Semi-regular assumptions only provide a rough heuristic.
- Introducing high degree equations might boost the rank, but is now even harder to analyze.
- Experiments are difficult to run due to sub-exponential blow-up in the size of the Macaulay matrix.

Recap and reflections

- **Recap**: we formulate a quadratic system of equations for LPN. Falsely assuming the Macaulay matrix is full rank suggests $O(\sqrt{m})$-expansion on this system is sufficient. Assuming semi-regularity suggests an upper bound of $O(m)$-expansion is required.
- Question: Is there some clever way to increase the rank of a Macaulay matrix at lower degrees of expansion?
- Question: We proposed adding random high degree equations that hold with high probability, but how does one analyze the rank of the matrix?
- Question: Is there a better system of equations for LPN over $\mathbb{F}_p$?

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