Quantum Advantage via Solving Multivariate Polynomials

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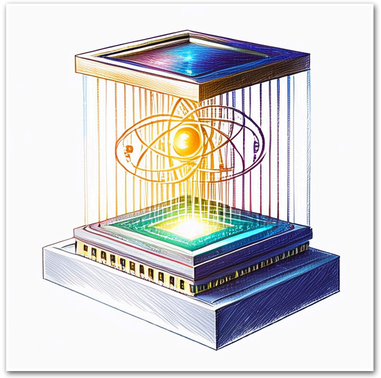
Quantum Advantage in **NP**-search



vs.

Are there **NP**-search (solutions are classically efficiently verifiable) problems that are

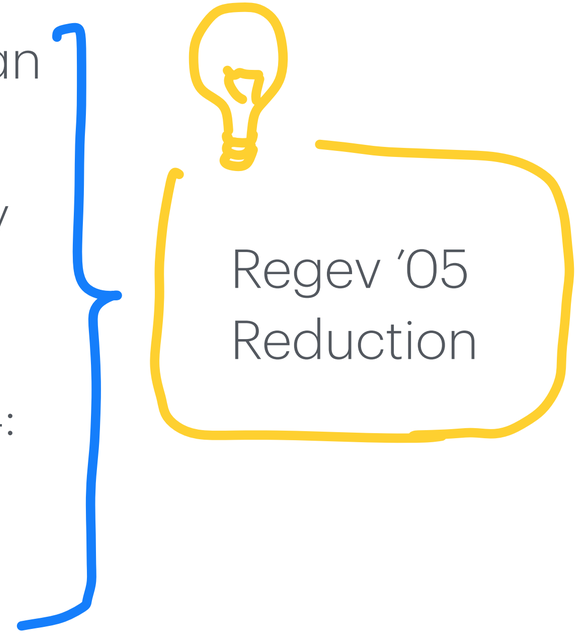
quantum polynomial-time tractable, yet **classically hard-to-solve**?

Quantum Advantage in **NP**-search

Quantumly easy, classically hard?

- **Hidden Subgroup Problems**, e.g. Simon's problem, Bernstein-Vazirani, factoring, discrete-log, Pell's equations, offer conjectured advantage from structured (*periodicity*) **NP**-search problems.

- **Yamakawa-Zhandry (YZ)** '22: Relative to *a random oracle,* there exists an *unstructured* **NP**-search problem that is quantumly easy-to-solve, yet unconditionally hard against any computationally-unbounded adversary making a polynomial number of classical queries.



- Jordan-Shutty-Wootters-Zalcman-Schmidhuber-King-Isakov-Babbush '24: **Decoded Quantum Interferometry (DQI)** speedups for optimization problems (e.g. optimal polynomial intersection).

Our Work: Sampling Roots for a Multivariate Polynomial System

Multivariate Polynomial Systems over Finite Fields

Is solving conjecturably quantumly and classically hard?

- **Worst-case complexity:** Deciding if a multivariate quadratic system over $\mathbb{F}_2$ has roots is **NP** -complete [Fraenkel-Yesha '77, Garey-Johnson '79].
- **Average-case complexity:** Sample *uniform random* $\mathscr{P} \triangleq \{p_i \leftarrow_R \mathbb{F}_2^{d\leq 2}[X_1, \ldots, X_n]\}_{i\in[m]}$. Given $(\mathscr{P}, \mathbf{y} = \mathscr{P}(\mathbf{x}))$, can you recover a preimage of $\mathbf{y}$?
  - Overdetermined regime: When $m = \Omega(n^2)$, polynomial time solvable via a linearization algorithm [Kipnis-Shamir '99]. Otherwise, believed to be $exp(n^2/m)$ hard.
  - Underdetermined regime: When $m = O(\sqrt{n})$, polynomial time solvable via [Section 7 of Kipnis-Patarin-Goubin '99]. Otherwise, believed to be $exp(m)$ hard.

Multivariate Polynomial Systems over Finite Fields

Is solving conjecturably quantumly and classically hard?

- **Quantum complexity:** No exponential speed up known, only polynomial speedups.
  - There's Grover search: $O(2^{n/2})$.
  - [Faugère-Kahrobaei-Kaplan-Kashefi-Perret '17] improve this to $O(2^{0.462n})$ quantum gates for the case of $m = n$ under some algebraic assumption on the system.

Structured Multivariate Polynomial Systems over Finite Fields

Cryptographers use these!

- **Mask easy-to-invert system by hidden linear transformation** [Patarin '96, '97, Kipnis-Patarin-Goubin '99]**.**
  - **Example** (Oil-&-Vinegar [Patarin '97, Kipnis-Patarin-Goubin '99]): Consider coefficient matrix of the form,

$$\mathbf{A} \triangleq \begin{pmatrix} \mathbf{0} & \mathbf{A}_1 \\ \mathbf{A}_2 & \mathbf{A}_3 \end{pmatrix} \in \mathbb{F}_q^{n \times n}.$$

$$\text{e.g. } \mathbf{x}^\top \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \mathbf{x} = 3x_1x_2 + 3x_2^2.$$

  - A random assignment to some fraction of the variables results in a linear system in the remaining variables.
  - Conjecturably, for appropriate parameters, it's hard-to-invert an underdetermined polynomial system $\{\mathbf{T}^\top \cdot \mathbf{A}^{(i)} \cdot \mathbf{T}\}_{i \in [m]}$. In fact, conjecturably quantum hard (in "post-quantum" signature competition).

Are there quantumly easy, yet conjecturably classically-hard, multivariate polynomial systems?

Recalling the YZ Proof of Quantumness

(results extend beyond $\mathbb{F}_2$, as do ours)

- For $i \in [n^2]$, let $H_i : \mathbb{F}_2^n \to \mathbb{F}_2$ be random functions.
- Let $C \subseteq \mathbb{F}_2^{n^3}$ be a code* and view a codeword $\mathbf{c} = \mathbf{c}_1 \| \mathbf{c}_2 \| \cdots \| \mathbf{c}_{n^2}$ where each $\mathbf{c}_i \in \mathbb{F}_2^n$.
- Define $H : C \to \mathbb{F}_2^{n^2}$ as follows:

$$H(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n^2}) = (H_1(\mathbf{c}_1), \ldots, H_{n^2}(\mathbf{c}_{n^2}))$$

- YZ shows unconditionally one-way against classical probabilistic algorithms in ROM with polynomially bounded queries to $(H_i)_{i \in [n^2]}$ and there exists an efficient quantum algorithm in the QROM, given any image value y, samples stat. close to uniformly from the set of preimages of y.

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Any instantiation of H_i gives insight into quantum easiness, e.g. LWE, LPN, your favorite problem…

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Natural thought: Can we instantiate H_i with random low-degree polynomials?

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Our Work: Instantiation with Random Low-degree Polynomials

Natural thought: Can we instantiate H_i with random low-degree polynomials?

Our work: Yes! For any $d \geq 3$, we can instantiate the H_i with uniform random at most degree d polynomials.

The resulting system is efficiently quantumly invertible, and conjecturably classically hard.

In fact, our results extend to any distribution over polynomials that is (1) shift-invariant and (2) 2-wise independent.

Our Polynomial System

- Fix $d \geq 3$. Total of n^3 variables, organized into n^2 blocks of n variables.

1. **Degree d constraints**: Sample n^2 many **random** at most degree d polynomials, $\{p_i\}_{i \in [n^2]}$, each on a disjoint block of variables.

2. **Linear constraints**: a Generalized Reed-Solomon parity-check matrix over the field extension $\mathbb{F}_{2^n}$:

$$\mathbf{H} \in \mathbb{F}_{2^n}^{(1-\alpha)n^2 \times n^2} \leftrightarrow \overline{\mathbf{H}} \in \mathbb{F}_2^{(1-\alpha)n^3 \times n^3}.$$

$$\overline{\mathbf{H}} \cdot \mathbf{x} = \mathbf{0}.$$

Our Polynomial System

$p_1(x_1, \ldots, x_n)$

$p_2(x_{n+1}, \ldots, x_{2n})$

$\ldots$

$p_{n^2}(x_{n^3-n+1}, \ldots, x_{n^3})$

} degree d

GRS parity check $H =$

$\mathbb{F}_{2^n}^*$ non-zero column multipliers

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_{n^2} \\ u_1 a_1 & u_2 a_2 & \cdots & u_{n^2} a_{n^2} \\ u_1 a_1^2 & u_2 a_2^2 & \cdots & u_{n^2} a_{n^2}^2 \\ \vdots & \vdots & & \vdots \\ u_1 a_1^{(1-\alpha)n^2-1} & u_2 a_2^{(1-\alpha)n^2-1} & \cdots & u_{n^2} a_{n^2}^{(1-\alpha)n^2-1} \end{bmatrix}$$

distinct in $\mathbb{F}_{2^n}$

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$p_1(x_1, \ldots, x_n)$

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$\ddots$

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} degree d

Converted to $\mathbb{F}_2$

$\overline{H} =$ ← $n \times n$ block matrix over $\mathbb{F}_2$

$$\begin{bmatrix} M(u_1) & M(u_2) & \cdots & M(u_{n^2}) \\ M(u_1 a_1) & M(u_2 a_2) & \cdots & M(u_{n^2} a_{n^2}) \\ M(u_1 a_1^2) & M(u_2 a_2^2) & \cdots & M(u_{n^2} a_{n^2}^2) \\ \vdots & \vdots & \ddots & \vdots \\ M(u_1 a_1^{(1-\alpha)n^2-1}) & & \cdots & M(u_{n^2} a_{n^2}^{(1-\alpha)n^2-1}) \end{bmatrix}$$

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$$\ell_1(x_1, \ldots, x_{n^3}) = \sum_{i=1}^{n^3} \alpha_i^{(1)} x_i$$

$$\ell_2(x_1, \ldots, x_{n^3}) = \sum_{i=1}^{n^3} \alpha_i^{(2)} x_i$$

$$\vdots \qquad \vdots$$

$$\ell_{(1-\alpha)n^3}(x_1, \ldots, x_{n^3}) = \sum_{i=1}^{n^3} \alpha_i^{((1-\alpha)n^3)} x_i$$

} $(1-\alpha)n^3$ Linearly indep. linear equations

Our Polynomial System

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$p_2(x_{n+1}, \ldots, x_{2n})$

$\ldots$

$p_{n^2}(x_{n^3-n+1}, \ldots, x_{n^3})$

} degree d

$x_1 = \sum_{i \neq 1} \alpha_i^{(1)} x_i$

$x_2 = \sum_{i \neq n+1} \alpha_i^{(2)} x_i$

$\vdots$

$x_{(1-\alpha)n^3} = \sum_{i \neq (1-\alpha)n^3}^{(1-\alpha)n^3} \alpha_i x_i$

} $(1-\alpha)n^3$ Linearly indep. linear equations

Each linear constraint allows for the backsubstitution for 1 variable.

Post-substitution, the degree d constraints are in αn^3 variables.

Classical Attacks

Why $d \geq 3$?

- There is a known normal form for degree two polynomials in characteristic two fields [Lidl-Niederreiter '97]:
  - Can be expressed in the form $z_1 z_2 + z_3 z_4 + \ldots + z_{r-1} z_r$ under some efficiently computable linear transformation on the variables $(x_1, \ldots, x_n)$ for some r dependent on the polynomial.
  - Recall, our degree d constraints are on disjoint variables, so for each polynomial constraint, fix $n/2$ of the variables to obtain a underdetermined linear system.

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Classical Attacks

Known Attack Strategies Fail

- Similar other attacks on underdetermined quadratic systems also add specific linear constraints to produce an affine system. E.g. [Thomae-Wolf '12, Furue-Nakamura-Takagi '21, Hashimoto '23]—**not known to extend to $d \geq 3$.**
  - $2^{-\Omega(n^2)}$ probability that for a random degree 3 polynomial, no choice of $n - o(n)$ affine equations that reduce it to a degree 2 polynomial.
  - Reduce #vars to $o(n^3)$ results in no solutions (code constraints give $O(n^3)$ equations).
- Other attacks:
  - Non-trivial exhaustive search exponential in n^2 time.
  - Gröbner basis algorithms experimentally exponential, but hard to determine exactly what exponential.

Analysis of the YZ Quantum Algorithm with Multivariate Polynomial Systems

The Prior Algorithmic Framework

[Yamakawa-Zhandry '22, Regev '05]

- A standard measurement of a quantum state

$$|\phi\rangle = \sum_{\mathbf{x}\in\mathbb{F}^n} V(\mathbf{x}) \cdot |\mathbf{x}\rangle$$

observes $\mathbf{x}$ with probability $|V(\mathbf{x})|^2$ where $V : \{0,1\}^n \to \mathbb{C}$.

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- Define

$$|\psi\rangle = \sum_{\mathbf{y}\in\mathbb{F}^N} W(\mathbf{y}) \cdot |\mathbf{y}\rangle.$$

- **Using $|\phi\rangle, |\psi\rangle$, can we produce their coordinate-wise product? i.e.**

$$\sum_{\mathbf{x}\in\mathbb{F}^N} (V \cdot W)(\mathbf{x}) \cdot |\mathbf{x}\rangle.$$

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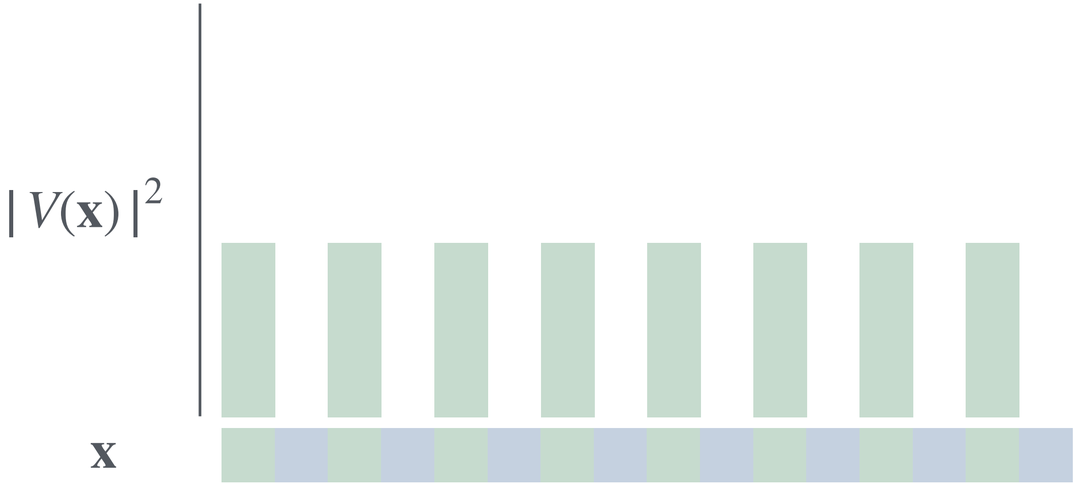
Their tensor contains undesired cross-terms:

$$|\phi\rangle|\psi\rangle = \sum_{\mathbf{x},\mathbf{y}\in\mathbb{F}^N} V(\mathbf{x})W(\mathbf{y}) \cdot |\mathbf{x}\rangle|\mathbf{y}\rangle.$$

The Coordinate-wise Product

- Let $|\phi\rangle$ be a uniform superposition over all codewords of the Generalized Reed-Solomon Code, so measuring this state results in a uniform random codeword, i.e.

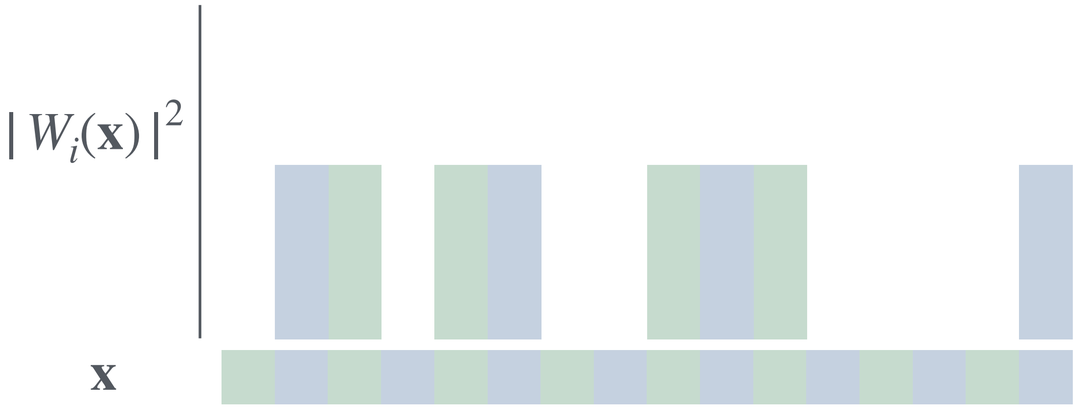
$$|\phi\rangle = \sum_{\mathbf{x}\in\mathbb{F}^N} V(\mathbf{x}) \cdot |\mathbf{x}\rangle, \text{ where } V(\mathbf{x}) = \begin{cases} 1/\sqrt{|C|} & \mathbf{x} \in C \\ 0 & \mathbf{x} \notin C \end{cases}.$$



The Coordinate-wise Product

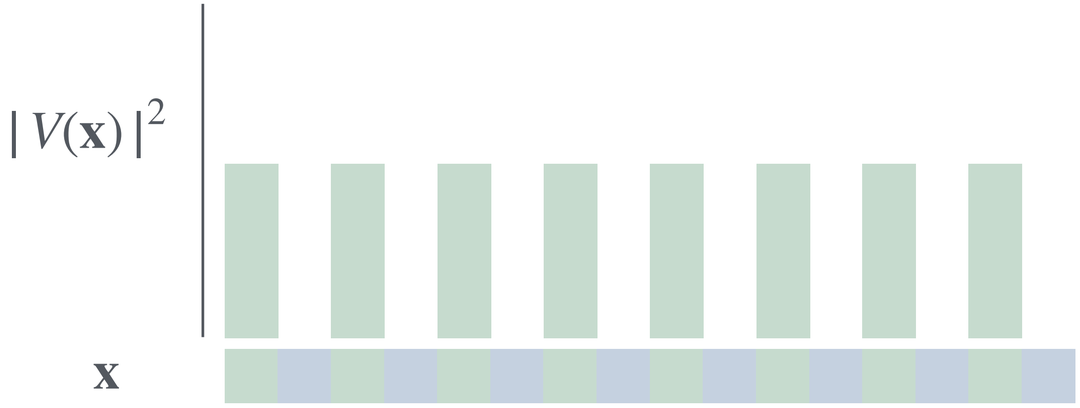
- Let $|\psi\rangle$ be a uniform superposition over all the roots of the degree d constraints, so measuring this state results in a uniform solution to the n^2 many polynomials p_i defined on disjoint variables. i.e. $|\psi\rangle = \bigotimes_{i=1}^{n^2} |\psi_i\rangle$ where for $i \in [n^2]$

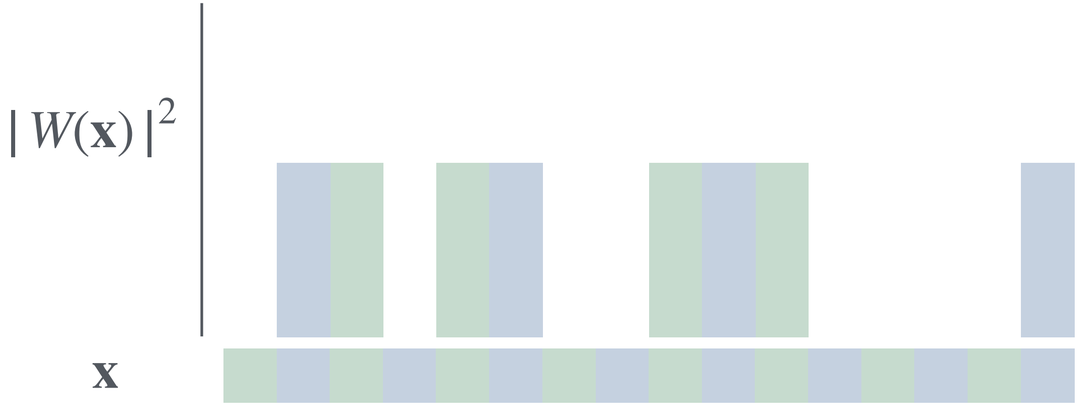
$$|\psi_i\rangle = \sum_{\mathbf{y} \in \mathbb{F}_2^n} W_i(\mathbf{y}) \cdot |\mathbf{y}\rangle \text{ where } W_i(\mathbf{y}) = \begin{cases} 1/\sqrt{|R_i|} & p_i(\mathbf{y}) = 0 \\ 0 & \text{o.w.} \end{cases}.$$



The Coordinate-wise Product

- Solving the polynomial system defined by the code constraint $\overline{\mathbf{H}} \cdot \mathbf{x} = \mathbf{0}$ and the random degree d polynomial system on disjoint variable blocks, $\{p_i\}_{i \in [n]}$, is exactly finding an $\mathbf{x}$ such that $V(\mathbf{x}) \neq 0$ AND $W(\mathbf{x}) \neq 0$.
- Therefore, **measuring the coordinate-wise product always gives us a solution to the polynomial system.**





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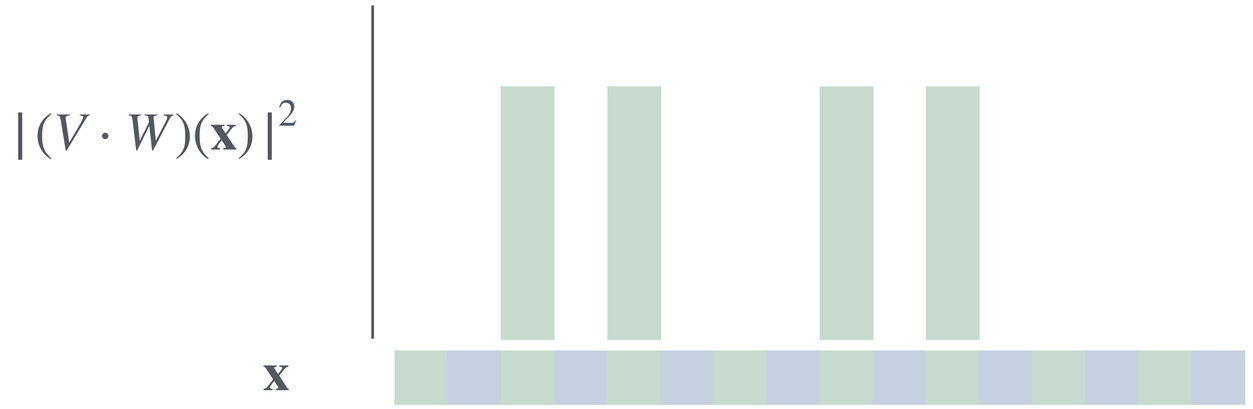

The Prior Algorithmic Framework

Coordinate-wise Product <—> Convolution [Yamakawa-Zhandry '22, Regev '05]

$$\sum_{\mathbf{z}\in\mathbb{F}^N} (\hat{V} * \hat{W})(\mathbf{z}) \,|\, \mathbf{z}\rangle = \mathsf{QFT} \sum_{\mathbf{z}\in\mathbb{F}^N} (V \cdot W)(\mathbf{z}) \,|\, \mathbf{z}\rangle$$

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- Apply a unitary addition to add the first register into the second register.

$$\mathbf{U}_{\mathsf{add}}\,|\hat{\phi}\rangle\,|\hat{\psi}\rangle = \sum_{\mathbf{x},\mathbf{y}\in\mathbb{F}^N} \hat{V}(\mathbf{x})\hat{W}(\mathbf{y}) \cdot |\mathbf{x}\rangle\,|\mathbf{x}+\mathbf{y}\rangle$$

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- **Wishful thinking:** If we could *uncompute* the first register, the resulting state would be

$$\sum_{\mathbf{z}\in\mathbb{F}^N} (\hat{V} * \hat{W})(\mathbf{z})\,|\mathbf{z}\rangle = \mathsf{QFT} \sum_{\mathbf{z}\in\mathbb{F}^N} (V \cdot W)(\mathbf{z})\,|\mathbf{z}\rangle$$

from which we could obtain our desired coordinate-wise product state by inverting the QFT.

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from which we could obtain our desired coordinate-wise product state by inverting the QFT.

Main Question:
How do you *uncompute* the first register?

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from which we could obtain our desired coordinate-wise product state by inverting the QFT.

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YZ'22: Treat $\mathbf{y}$ as noise and decode a noisy codeword $\mathbf{x}+\mathbf{y}$.

The Quantum Algorithm

1. Prepare uniform superposition over codewords $|\phi\rangle = \sum_{\mathbf{x}\in\mathbb{F}^N} V(\mathbf{x}) \cdot |\mathbf{x}\rangle$ and over roots of each polynomial $|\psi_i\rangle = \sum_{\mathbf{y}\in\mathbb{F}_2^n} W_i(\mathbf{y}) \cdot |\mathbf{y}\rangle$ for $i \in [n^2]$. Let $|\psi\rangle = \otimes_i |\psi_i\rangle$.

2. Compute $\left(\left(I \otimes \mathrm{QFT}^{-1}\right) \circ \mathbf{U}_{\mathsf{Decode}} \circ \mathbf{U}_{\mathsf{add}}\right)(\mathrm{QFT}|\phi\rangle \otimes \mathrm{QFT}|\psi\rangle)$.

3. Output measurement of the second register.

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3. Output measurement of the second register.

Two Technical Challenges

We have that

$$\left(\mathbf{U}_{\mathsf{Decode}} \circ \mathbf{U}_{\mathsf{add}}\right)(\mathsf{QFT}\,|\phi\rangle \otimes \mathsf{QFT}\,|\psi\rangle) = \sum_{\mathbf{x},\mathbf{y}\in\mathbb{F}^N} \hat{V}(\mathbf{x})\hat{W}(\mathbf{y}) \cdot |\mathbf{x} - \mathsf{Decode}_{\mathbf{C}^\perp}(\mathbf{x}+\mathbf{y})\rangle\,|\mathbf{x}+\mathbf{y}\rangle$$

Crux: We want to show decoding almost always succeeds!

1. For what average-case error distributions can we *uniquely* decode?

2. What error distribution is induced by a uniform distribution over the root set of multivariate polynomials over disjoint variables?

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Crux: We want to show decoding almost always succeeds!

1. For what average-case error distributions can we *uniquely* decode?

Generic coding question.

2. What error distribution is induced by a uniform distribution over the root set of multivariate polynomials over disjoint variables?

Specific to our polynomial system.

$$\sum_{\mathbf{x},\mathbf{y}\in\mathbb{F}^N} \hat{V}(\mathbf{x})\hat{W}(\mathbf{y}) \cdot |\mathbf{x} - \mathrm{Decode}_{\mathbf{C}^\perp}(\mathbf{x}+\mathbf{y})\rangle |\mathbf{x}+\mathbf{y}\rangle \approx \sum_{\mathbf{x},\mathbf{y},\ \text{decodable}} \hat{V}(\mathbf{x})\hat{W}(\mathbf{y}) \cdot |\mathbf{0}\rangle |\mathbf{x}+\mathbf{y}\rangle$$

$$\approx |\mathbf{0}\rangle \otimes \mathrm{QFT} \sum_{\mathbf{z}\in\mathbb{F}^N} (V \cdot W)(\mathbf{z}) |\mathbf{z}\rangle$$

$$\mathscr{B} \triangleq \{\mathbf{e} \in \mathbb{F}_{2^n}^{n^3} : \exists \mathbf{x} \in C^\perp,\ \mathrm{Decode}_{C^\perp}(\mathbf{x}+\mathbf{e}) \neq \mathbf{x}\}$$

$$\sum_{\mathbf{e}\in\mathscr{B}} \left|\hat{W}(\mathbf{e})\right|^2 \leq \mathrm{negl}(n)$$

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probability mass on a bad error e.

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? Δ-inequality ?

$= \begin{cases} |C^\perp|^{-1/2} & \text{if } \mathbf{x}\in C \\ 0 & \text{o.w.} \end{cases}$

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e.g. $|a|^2 + |b|^2 \lesssim |a+b|^2$

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$\Longrightarrow$ + shift-invariance

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Current focus. Proof in two step process...

1. Uniquely Decodable Error Distributions

Folded Reed-Solomon Codes

YZ'22: Burst error distributions with the following property are uniquely decodable:

$$\text{for all } i \in [n^2],\ \mathbf{e}_i = \begin{cases} \mathbf{0} & \text{w.p. } 1/2 \\ \mathsf{Unif}(\mathbb{F}_2^n \backslash \mathbf{0}) & \text{w.p. } 1/2 \end{cases}.$$

No reason that a root set of uniform random degree d polynomials would induce such an error distribution.

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Our work: observes the YZ proof of the above extends to *any* burst error distributions where blocks are $\mathbf{0}$ w.p. $1/2$, and takes on any point in $\mathbb{F}_2^n \backslash \{\mathbf{0}\}$ w.p. $2^{-\Omega(n)}$ i.e. high min-entropy.

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The question remains: does our uniform random polynomial distribution induce such a distribution?

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Our work: observes the YZ proof of the above extends to *any* burst error distributions where blocks $\mathbf{0}$ are probability $1/2$, and have probability mass $2^{-\Omega(n)}$ on any point in $\mathbb{F}_2^n \backslash \{\mathbf{0}\}$, i.e. high min-entropy.

Yes! In fact, holds for any 2-wise indep. distribution on polynomials.

2. Distribution Induced by Root Sets are Uniquely Decodable

$$|\psi_i\rangle = \sum_{\mathbf{y}\in\mathbb{F}_2^n} W_i(\mathbf{y}) \cdot |\mathbf{y}\rangle \text{ where } W_i(\mathbf{y}) = \begin{cases} 1/\sqrt{|R_i|} & p_i(\mathbf{y}) = 0 \\ 0 & \text{o.w.} \end{cases}.$$

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What is the distribution over $\mathbb{F}_2^n$ defined by the probability mass function: $\mathbb{E}_{p_i}\left[\|\hat{W}_i(\,\cdot\,)\|^2\right]$?

(why? By Markov, with all but negl. prob. over p_i, almost all mass on GOOD)

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Easy observation: $\mathbb{E}_{p_i}\left[\|\hat{W}_i(\mathbf{0})\|^2\right] = 2^{-n} \cdot \mathbb{E}_{p_i}\left[|R_i|\right] = \frac{1}{2}.$

Due to the 1-wise independence of random inhomogeneous degree d polynomials:

$$\forall \mathbf{y} \in \mathbb{F}_2^n, \text{ we have } \mathbb{E}_{p_i}[\mathbb{I}_{\mathbf{y}}(p_i)] = 1/2.$$

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2. Distribution Induced by Root Sets are Uniquely Decodable

For all $\mathbf{y} \in \mathbb{F}_2^n \backslash \{\mathbf{0}\}, n \geq 10$, we can show that

$$\mathbb{E}_p \left[\|\hat{W}_i(\mathbf{y})\|^2 \right] \leq 2^{-n/2} \text{ (Property 2) .}$$

2-wise independence of random inhomogeneous degree d polynomials => Property 2.

$\forall \mathbf{x} \neq \mathbf{y} \in \mathbb{F}_2^n$, we have $\mathbb{E}_{p_i}[\mathbb{I}_{\mathbf{x}}(p_i) \cdot \mathbb{I}_{\mathbf{y}}(p_i)] = 1/4$.

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For all $\mathbf{y} \in \mathbb{F}_2^n \backslash \{\mathbf{0}\}, n \geq 10$, we can show that

$$\mathbb{E}_p \left[\|\hat{W}_i(\mathbf{y})\|^2 \right] \leq 2^{-n/2} \text{ (Property 2) .}$$

To do so, we can express the Fourier coefficient for all $\mathbf{y} \neq \mathbf{0}$ as

$$\hat{W}_i(\mathbf{y}) = 2^{-n/2} |R_i|^{-1/2} \cdot \sum_{\mathbf{z} \in \mathbb{F}_2^n} (-1)^{p_i(\mathbf{z}) + \langle \mathbf{y}', \mathbf{z} \rangle_2}$$

Effectively, need to consider the behavior of the root set of p_i shifted by linear polynomials.

2. Distribution Induced by Root Sets are Uniquely Decodable

$$\mathbb{E}_p\left[\|\hat{W}_i(\mathbf{0})\|^2\right] = \frac{1}{2} \text{ (Property 1)}$$

For all $\mathbf{y} \in \mathbb{F}_2^n \backslash \{\mathbf{0}\}, n \geq 10$, we have $\mathbb{E}_p\left[\|\hat{W}_i(\mathbf{y})\|^2\right] \leq 2^{-n/2}$ (Property 2).

Together, satisfy unique decodability from step 1.

$\therefore$ **Any** 2-wise independent distribution on $\mathbb{F}_2[x_1, \ldots, x_n]$ gives the above distribution.

Using Shift-invariance

$$\sum_{\mathbf{x},\mathbf{y}\in\mathbb{F}^N} \hat{V}(\mathbf{x})\hat{W}(\mathbf{y}) \cdot |\mathbf{x} - \mathsf{Decode}_{\mathbf{C}^\perp}(\mathbf{x}+\mathbf{y})\rangle |\mathbf{x}+\mathbf{y}\rangle \overset{(1)}{\approx} \sum_{\mathbf{x},\mathbf{y},\ \text{decodable}} \hat{V}(\mathbf{x})\hat{W}(\mathbf{y}) \cdot |\mathbf{0}\rangle |\mathbf{x}+\mathbf{y}\rangle$$

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Using Shift-invariance

Exploiting symmetry!

Want to show: $$\sum_{\mathbf{z}\in\mathbb{F}_2^n} \left| \sum_{(\mathbf{x},\mathbf{z}-\mathbf{x})\in \mathsf{BAD}} \hat{V}(\mathbf{x})\hat{W}(\mathbf{z}-\mathbf{x}) \right|^2 \leq \mathsf{negl}(n)$$

Yamakawa-Zhandry: Uses some permutation-invariant property of random oracles.

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Yamakawa-Zhandry: Uses some permutation-invariant property of random oracles.

We use **shift-invariance**:

A distribution $\mathscr{D}$ is **shift-invariant** if for all $\mathbf{s}\in\mathbb{F}_2^n$, the distribution $\mathscr{D}_{\mathbf{s}}$ where you sample a random $p\sim\mathscr{D}$ and then output $p_{\mathbf{s}}$ where $p_{\mathbf{s}}(\mathbf{x})=p(\mathbf{x}+\mathbf{s})$ is distributed identically to $\mathscr{D}$.

Concluding thoughts

- The quantum algorithm works for any shift-invariant, and 2-wise independent distribution over polynomials. Two examples:
  - Uniform random degree bounded polynomials for any $d \geq 2$.
  - Random high (poly in n) degree d, sparse (nonzero w.p. $1/n^{d-1}$) polynomials.
- **Recap:** First evidence of an efficiently quantumly invertible yet plausibly classically hard-to-invert polynomial system, and a non-oracle low-degree instantiation of YZ.

Further Thoughts

- Connections to complement sampling: Thanks to an anonymous SODA '26 reviewer who pointed out a simpler proof for correctness in the case of $\mathbb{F}_2$ using complement sampling [Benedetti-Buhrman-Weggemans '25].
  - Our results extend to larger finite fields, whereas this argument does not immediately extend (complement set of $H^{-1}(1)$ would no longer be $H^{-1}(0)$).
- Further classical cryptanalysis on this polynomial system.
- **Main question for future work:** Can we find other interesting instantiations of the R.O.?

Thank you!